

Flow: Waves and instabilities in stationary plasmas

Overview

- **Introduction:** theoretical themes for a complete MHD description of laboratory and astrophysical plasmas, static versus stationary plasmas;
- **Spectral theory of stationary plasmas:** Frieman–Rotenberg formalism for waves and instabilities, quadratic eigenvalue problem gives complex eigenvalues, implications of the Doppler shift for the continuous spectra;
- **Kelvin–Helmholtz instability of streaming plasmas:** gravitating plasma with an interface where the velocity changes discontinuously, influence of the magnetic field;
- **Magneto-rotational instability of rotating plasmas:** derivation of the dispersion equation, growth rates of instabilities, application to accretion disks.

Theoretical themes

- In overview of magnetic structures and dynamics (Chap. 8), we encountered:
 - Central importance of magnetic flux tubes \Rightarrow **Cylindrical plasmas**, 1D: $f(r)$
[Volume 1: Chap. 9]
 - Astrophysical flows (winds, disks, jets) \Rightarrow **Plasmas with background flow**
[this lecture, Volume 2: MHDF.pdf]
 - Explosive phenomena due to reconnection \Rightarrow **Resistive MHD**
[Volume 2: MHDR.pdf]
 - Magnetic confinement for fusion (tokamak) \Rightarrow **Toroidal plasmas**, 2D: $f(r, \vartheta)$
[Volume 2: MHDT.pdf]
 - Shocks, transonic flows, dynamos, turbulence \Rightarrow **Nonlinear MHD**
[Volume 2: MHDS.pdf]
 - All plasma dynamics (e.g. space weather) \Rightarrow **Computational MHD**
[Volume 2: ...]
- **MHD with background flow** is the most urgent topic (also for fusion research since divertors and neutral beam injection cause significant flows in tokamaks).
 - \Rightarrow **From static ($v = 0$) to stationary ($v \neq 0$) plasmas!**

Static versus stationary plasmas

- Starting point is the *set of nonlinear ideal MHD equations*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p - \mathbf{j} \times \mathbf{B} - \rho \mathbf{g} = 0, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad (2)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (4)$$

with gravitational acceleration $\mathbf{g} = -\nabla \Phi_{\text{gr}}$ due to external gravity field Φ_{gr} .

- Recall the simplicity of **static equilibria** ($\partial/\partial t = 0, \mathbf{v} = 0$),

$$\nabla p = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad (5)$$

with perturbations described by **self-adjoint operator \mathbf{F} with real eigenvalues ω^2** :

$$\mathbf{F}(\boldsymbol{\xi}) = \rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \Rightarrow \mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho \omega^2 \hat{\boldsymbol{\xi}}. \quad (6)$$

- Can one construct a similar powerful scheme for stationary plasmas ($\mathbf{v} \neq 0$)?**

Stationary equilibria

- Basic nonlinear ideal MHD equations for **stationary equilibria** ($\partial/\partial t = 0$):

$$\nabla \cdot (\rho \mathbf{v}) = 0, \quad (7)$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}, \quad \mathbf{j} = \nabla \times \mathbf{B}, \quad (8)$$

$$\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \quad (9)$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (10)$$

⇒ None of them trivially satisfied now (except for simple geometries)!

- For *plane gravitating plasma slab*, equilibrium unchanged w.r.t. static case:

$$(p + \frac{1}{2}B^2)' = -\rho g \quad (' \equiv d/dx). \quad (11)$$

- For *cylindrical plasma*, the equilibrium is changed significantly by the *centrifugal acceleration*, $-\mathbf{v} \cdot \nabla \mathbf{v} = (v_\theta^2/r)\mathbf{e}_r$:

$$(p + \frac{1}{2}B^2)' + \frac{1}{r}B_\theta^2 = \frac{1}{r}\rho v_\theta^2 - \rho \Phi'_{\text{gr}} \quad (' \equiv d/dr). \quad (12)$$

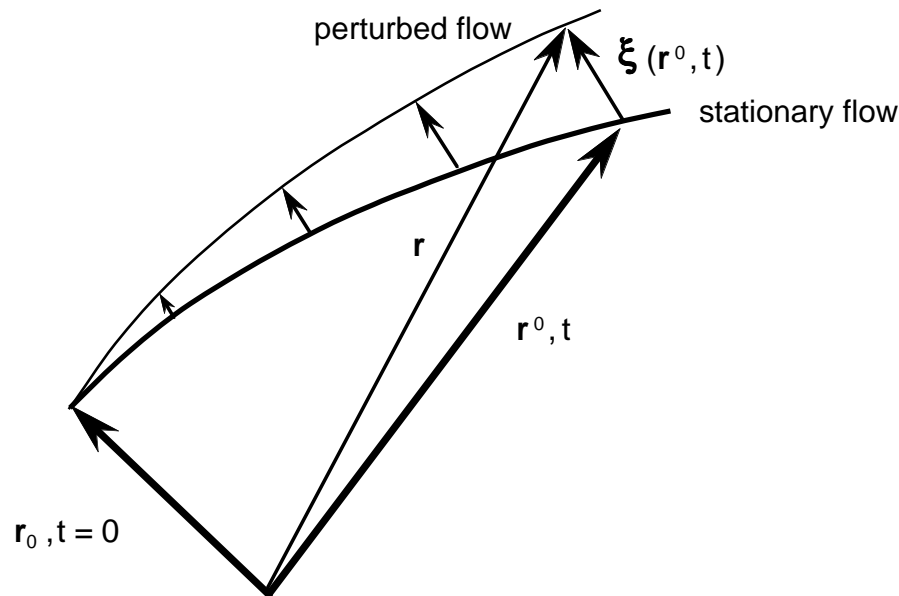
⇒ Modifications for plane and cylindrical stationary flows quite different:

translations and rotations are physically different phenomena.

Frieman–Rotenberg formalism

 [Rev. Mod. Phys. **32**, 898 (1960)]

Spectral theory for general stationary equilibria (no further simplifying assumptions):



- First, construct **displacement vector $\boldsymbol{\xi}$** connecting perturbed flow at position \mathbf{r} with unperturbed flow at position \mathbf{r}^0 :

$$\mathbf{r}(\mathbf{r}^0, t) = \mathbf{r}^0 + \boldsymbol{\xi}(\mathbf{r}^0, t). \quad (13)$$

- **In terms of the coordinates (\mathbf{r}^0, t) , the equilibrium is time-independent:**

$$\rho = \rho^0(\mathbf{r}^0), \text{ etc., satisfying (7)–(10).}$$

- Gradient $\nabla = (\nabla \mathbf{r}^0) \cdot \nabla^0 = \nabla(\mathbf{r} - \boldsymbol{\xi}) \cdot \nabla^0 \approx \nabla^0 - (\nabla^0 \boldsymbol{\xi}) \cdot \nabla^0,$ (14)

and Lagrangian time derivative $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} \Big|_{\mathbf{r}^0} + \mathbf{v}^0 \cdot \nabla^0,$ (15)

yield expression for the velocity at the perturbed trajectory:

$$\mathbf{v}(\mathbf{r}^0 + \boldsymbol{\xi}, t) \equiv \frac{D\mathbf{r}}{Dt} = \frac{D\mathbf{r}^0}{Dt} + \frac{D\boldsymbol{\xi}}{Dt} = \mathbf{v}^0 + \mathbf{v}^0 \cdot \nabla^0 \boldsymbol{\xi} + \frac{\partial \boldsymbol{\xi}}{\partial t}. \quad (16)$$

Frieman–Rotenberg formalism (cont'd)

- Linearization of Eqs. (1), (3), (4) gives perturbed quantities in terms of ξ alone:

$$\rho \approx \rho^0 - \rho^0 \nabla^0 \cdot \xi = \rho^0 + \rho_E^1 + \xi \cdot \nabla^0 \rho^0, \quad (17)$$

$$p \approx p^0 - \gamma p^0 \nabla^0 \cdot \xi = p^0 + \pi + \xi \cdot \nabla^0 p^0, \quad (\pi \equiv p_E^1) \quad (18)$$

$$\mathbf{B} \approx \mathbf{B}^0 + \mathbf{B}^0 \cdot \nabla^0 \xi - \mathbf{B}^0 \nabla^0 \cdot \xi = \mathbf{B}^0 + \mathbf{Q} + \xi \cdot \nabla^0 \mathbf{B}^0, \quad (\mathbf{Q} \equiv \mathbf{B}_E^1) \quad (19)$$

where we will now drop the superscripts ⁰ (since everything has this superscript).

- Substitution in Eq. (2) (+ algebra!) \Rightarrow **spectral equation for equilibria with flow:**

$$\rho \frac{\partial^2 \xi}{\partial t^2} + 2\rho \mathbf{v} \cdot \nabla \frac{\partial \xi}{\partial t} - \mathbf{G}(\xi) = 0, \quad (20)$$

$$\mathbf{G} \equiv \mathbf{F} + \nabla \cdot (\xi \rho \mathbf{v} \cdot \nabla \mathbf{v} - \rho \mathbf{v} \mathbf{v} \cdot \nabla \xi), \quad (21)$$

$$\mathbf{F} \equiv -\nabla \pi - \mathbf{B} \times (\nabla \times \mathbf{Q}) + (\nabla \times \mathbf{B}) \times \mathbf{Q} + (\nabla \Phi) \nabla \cdot (\rho \xi).$$

- For normal modes, $\xi \sim \exp(-i\omega t)$, a quadratic eigenvalue problem is obtained:

$$\mathbf{G}(\xi) + 2i\rho\omega \mathbf{v} \cdot \nabla \xi + \rho\omega^2 \xi = 0, \quad (22)$$

where the **generalised force operator \mathbf{G} is selfadjoint (like \mathbf{F} in the static case) but eigenvalues ω are complex because of the Doppler shift operator $i\mathbf{v} \cdot \nabla$.**

Spectral equation for plane slab

- For the plane slab model of Chapter 7, extended with a *plane shear flow field*,

$$\begin{aligned}\mathbf{B} &= B_y(x)\mathbf{e}_y + B_z(x)\mathbf{e}_z, \quad \rho = \rho(x), \quad p = p(x), \\ \mathbf{v} &= v_y(x)\mathbf{e}_y + v_z(x)\mathbf{e}_z,\end{aligned}\tag{23}$$

the equilibrium is unchanged and the two new terms in \mathbf{G} yield: $\mathbf{v} \cdot \nabla \mathbf{v} = 0$ and $-\nabla \cdot (\rho \mathbf{v} \mathbf{v} \cdot \nabla \boldsymbol{\xi}) = -\rho (\mathbf{v} \cdot \nabla)^2 \boldsymbol{\xi}$, so that the eigenvalue problem (22) becomes:

$$\begin{aligned}\mathbf{G}(\boldsymbol{\xi}) + 2i\rho\omega\mathbf{v} \cdot \nabla \boldsymbol{\xi} + \rho\omega^2\boldsymbol{\xi} &= \mathbf{F}(\boldsymbol{\xi}) + \rho(\omega + i\mathbf{v} \cdot \nabla)^2 \boldsymbol{\xi} = 0 \\ \Rightarrow \mathbf{F}(\boldsymbol{\xi}) &= -\rho\tilde{\omega}^2 \boldsymbol{\xi}, \quad \tilde{\omega} \equiv \omega + i\mathbf{v} \cdot \nabla.\end{aligned}\tag{24}$$

- Hence, the equations for the static slab remain valid with the replacement

$$\omega \rightarrow \tilde{\omega}(x) \equiv \omega - \Omega_0(x), \quad \Omega_0 \equiv \mathbf{k}_0 \cdot \mathbf{v}(x),\tag{25}$$

where $\Omega_0(x)$ is the local Doppler shift and $\tilde{\omega}(x)$ is the *local Doppler shifted frequency* observed in a local frame co-moving with the plasma at the vertical position x .

- Since $\tilde{\omega}$ depends on x , eigenvalues will be shifted by some average of $\Omega_0(x)$ across the layer. If a static equilibrium is unstable (eigenvalue on positive imaginary axis), for the corresponding equilibrium with flow that eigenvalue moves into the complex plane and becomes an *overstable mode*.

HD continua for plane shear flow

- How are the MHD waves affected by background flow of the plasma?

First, consider **continuous spectrum in the HD case** for plane slab geometry, inhomogeneous fluid with horizontal flow:

$$\mathbf{v} = v_y(x)\mathbf{e}_y + v_z(x)\mathbf{e}_z.$$

- Lagrangian time derivative:

$$(Df/dt)_1 \equiv (\partial f/\partial t + \mathbf{v} \cdot \nabla f)_1 = -i\tilde{\omega}f_1 + f_0'v_{1x}, \quad \tilde{\omega} \equiv \omega - \mathbf{k}_0 \cdot \mathbf{v},$$

$\tilde{\omega}(x)$: frequency observed in local frame co-moving with fluid layer at position x .

- Singularities when $\tilde{\omega} = 0$ somewhere in the fluid \Rightarrow **HD flow continuum** $\{\Omega_0(x)\}$, consisting of the zeros of the local Doppler shifted frequency

$$\tilde{\omega} \equiv \omega - \Omega_0(x), \quad \Omega_0 \equiv -i\mathbf{v} \cdot \nabla = \mathbf{k}_0 \cdot \mathbf{v}, \quad \text{on the interval } x_1 \leq x \leq x_2.$$

These have been extensively investigated in the hydrodynamics literature.

[Lin 1955; Case 1960; Drazin and Reid, Hydrodynamic Stability (Cambridge, 2004)]

MHD continua for plane shear flow

- **Forward (+) / backward (−) Alfvén and slow continua, and fast cluster points:**

$$\Omega_A^\pm \equiv \Omega_0 \pm \omega_A, \quad \omega_A \equiv F/\sqrt{\rho}, \quad F \equiv -i \mathbf{B} \cdot \nabla = \mathbf{k}_0 \cdot \mathbf{B},$$

$$\Omega_S^\pm \equiv \Omega_0 \pm \omega_S, \quad \omega_S \equiv \sqrt{\frac{\gamma p}{\gamma p + B^2}} F/\sqrt{\rho}, \quad \Omega_0 \equiv -i \mathbf{v} \cdot \nabla = \mathbf{k}_0 \cdot \mathbf{v},$$

$$\Omega_F^\pm \equiv \pm\infty.$$

- The flow contribution to the MHD continua creates the following ordering of the local frequencies **(which are all real)** in the co-moving frame:

$$\Omega_F^- \leq \Omega_{f0}^- \leq \Omega_A^- \leq \Omega_{s0}^- \leq \Omega_S^- \leq \Omega_0 \leq \Omega_S^+ \leq \Omega_{s0}^+ \leq \Omega_A^+ \leq \Omega_{f0}^+ \leq \Omega_F^+.$$

- **The discrete spectra are monotonic for real ω outside these frequencies.**
- In the limit $\mathbf{B} \rightarrow 0$, the Alfvén and slow continua collapse into the flow continuum,

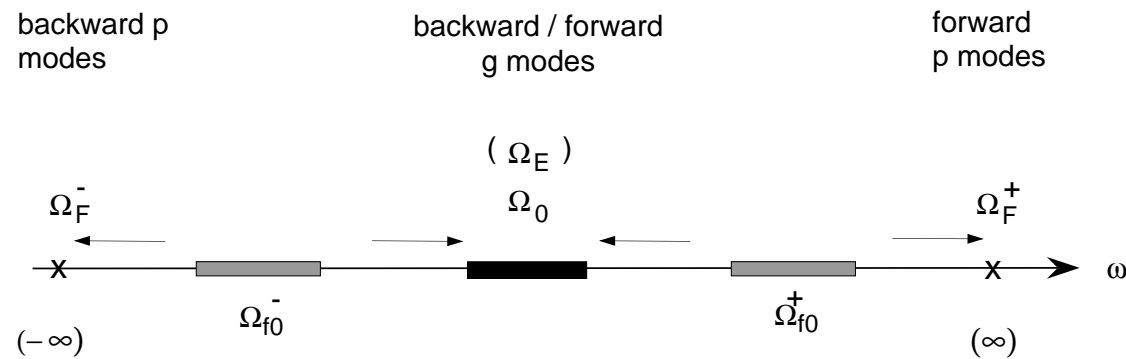
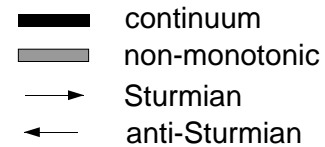
$$\Omega_A^\pm \rightarrow \Omega_0, \quad \Omega_S^\pm \rightarrow \Omega_0 \quad (\text{whereas } \Omega_F^\pm \text{ remains at } \pm\infty).$$

Vice versa, the HD flow continuum is absorbed by the MHD continua when $B \neq 0$. Hence, contrary to the literature, **there is no separate flow continuum in MHD!**

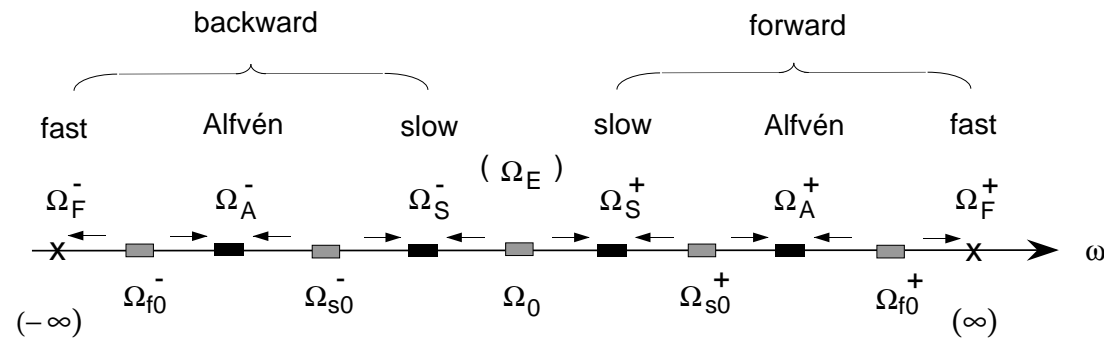
[Goedbloed, Beliën, van der Holst, Keppens, Phys. Plasmas **11**, 4332 (2004)]

Real parts of HD & MHD spectra

- **HD spectrum** of fluid flow:



- **MHD spectrum** of plasma flow:



Kelvin–Helmholtz instability: equilibrium

- Extend Rayleigh–Taylor instability of plasma–vacuum interface (sheets 6-36 – 6-42) to **plasma–plasma interface with two different velocities** (see figure on 6-36): *Rayleigh–Taylor + Kelvin–Helmholtz!*

- Upper layer ($0 < x \leq a$):

$$\begin{aligned} \rho &= \text{const}, \quad \mathbf{v} = (0, v_y, v_z) = \text{const}, \quad \mathbf{B} = (0, B_y, B_z) = \text{const}, \\ p' &= -\rho g \quad \Rightarrow \quad p = p_0 - \rho g x \quad (p_0 \geq \rho g a). \end{aligned} \quad (26)$$

Lower layer ($-b \leq x < 0$):

$$\begin{aligned} \hat{\rho} &= \text{const}, \quad \hat{\mathbf{v}} = (0, \hat{v}_y, \hat{v}_z) = \text{const}, \quad \hat{\mathbf{B}} = (0, \hat{B}_y, \hat{B}_z) = \text{const}, \\ \hat{p}' &= -\hat{\rho} g \quad \Rightarrow \quad \hat{p} = \hat{p}_0 - \hat{\rho} g x. \end{aligned} \quad (27)$$

- Jumps at the interface ($x = 0$):

$$\text{BC: } p_0 + \frac{1}{2} B_0^2 = \hat{p}_0 + \frac{1}{2} \hat{B}_0^2 \quad (\text{pressure balance}), \quad (28)$$

$$\Rightarrow \mathbf{j}^* = \mathbf{n} \times [[\mathbf{B}]] = \mathbf{e}_x \times (\mathbf{B} - \hat{\mathbf{B}}) \quad (\text{surface current}), \quad (29)$$

$$\Rightarrow \boldsymbol{\omega}^* = \mathbf{n} \times [[\mathbf{v}]] = \mathbf{e}_x \times (\mathbf{v} - \hat{\mathbf{v}}) \quad (\text{surface vorticity}).$$

Kelvin–Helmholtz instability: normal modes

- Now (in contrast to energy principle analysis of 6-36 – 6-42), **normal mode analysis:**

$$\xi \sim \exp [i(k_y y + k_z z - \omega t)].$$

- For *incompressible plasma*, taking limit $c^2 \equiv \gamma p / \rho \rightarrow \infty$ of Eq. (50) on sheet 7-20, *with plane flow*, replacing $\omega \rightarrow \tilde{\omega}$ (Eq. (25) of sheet F-7), basic ODE becomes:

$$\frac{d}{dx} \left[\rho(\tilde{\omega}^2 - \omega_A^2) \frac{d\xi}{dx} \right] - k_0^2 \left[\rho(\tilde{\omega}^2 - \omega_A^2) + \rho'g \right] \xi = 0. \quad (30)$$

Doppler shifted freq. $\tilde{\omega} \equiv \omega - \Omega_0$, $\Omega_0 \equiv \mathbf{k}_0 \cdot \mathbf{v}$; Alfvén freq. $\omega_A \equiv \mathbf{k}_0 \cdot \mathbf{B} / \sqrt{\rho_0}$.

- In this case, all equilibrium quantities constant so that ODEs simplify to equations with constant coefficients:

$$\xi'' - k_0^2 \xi = 0, \quad \text{BC } \xi(a) = 0 \quad \Rightarrow \quad \xi = C \sinh [k_0(a - x)], \quad (31)$$

$$\hat{\xi}'' - k_0^2 \hat{\xi} = 0, \quad \text{BC } \hat{\xi}(-b) = 0 \quad \Rightarrow \quad \hat{\xi} = \hat{C} \sinh [k_0(x + b)]. \quad (32)$$

\Rightarrow **Surface modes** (cusp-shaped eigenfunctions). This part is trivial, contains hardly any physics. Physics comes from the **BCs at $x = 0$ determining the eigenvalues**.

Kelvin–Helmholtz instability: interface conditions

- Now (in contrast to energy principle analysis of 6-36 – 6-42), *need both interface conditions* (model II* BCs), to determine relative amplitude \hat{C}/C and eigenvalue ω :
 - *First interface condition* (continuity of normal velocity):

$$[\mathbf{n} \cdot \boldsymbol{\xi}] = 0 \quad \Rightarrow \quad \xi(0) = \hat{\xi}(0) = 0 \quad \Rightarrow \quad C \sinh k_0 a = \hat{C} \sinh k_0 b. \quad (33)$$

- *Second interface condition* (pressure balance):

$$[\Pi + \mathbf{n} \cdot \boldsymbol{\xi} \mathbf{n} \cdot \nabla(p + \frac{1}{2}B^2)] = 0, \quad \Pi \equiv -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p + \mathbf{B} \cdot \mathbf{Q}, \quad (34)$$

where $\gamma p \nabla \cdot \boldsymbol{\xi}$ is undetermined. Determine Π from expression for compressible plasmas, [Book, Eq. \(7.99\)](#), with ω replaced by $\tilde{\omega}$ and taking limit $\gamma \rightarrow \infty$:

$$\Pi \equiv -\frac{N}{D} \xi' + \rho g \frac{\tilde{\omega}^2(\tilde{\omega}^2 - \omega_A^2)}{D} \xi \rightarrow \frac{\rho}{k_0^2}(\tilde{\omega}^2 - \omega_A^2)\xi'. \quad (35)$$

- Dividing the second by the first interface condition then gives

$$\left[\frac{\rho}{k_0^2}(\tilde{\omega}^2 - \omega_A^2) \frac{\xi'}{\xi} - \rho g \right] = 0 \quad \Rightarrow \quad \text{eigenvalue } \omega. \quad (36)$$

Kelvin–Helmholtz instability: dispersion equation

- Inserting solutions (31) and (32) for ξ and $\hat{\xi}$ yields the **dispersion equation**:

$$-\rho [(\omega - \Omega_0)^2 - \omega_A^2] \coth(k_0 a) - k_0 \rho g = \hat{\rho} [(\omega - \hat{\Omega}_0)^2 - \hat{\omega}_A^2] \coth(k_0 b) - k_0 \hat{\rho} g. \quad (37)$$

Describes magnetic field line bending (Alfvén), gravity (RT), velocity difference (KH).

- Approximations for long wavelengths ($k_0 x \ll 1$): $\coth k_0 x \approx (k_0 x)^{-1}$,
short wavelengths ($k_0 x \gg 1$): $\coth k_0 x \approx 1$.
- Solutions for short wavelengths (walls effectively at ∞ and $-\infty$):

$$\omega = \frac{\rho \Omega_0 + \hat{\rho} \hat{\Omega}_0}{\rho + \hat{\rho}} \pm \sqrt{-\frac{\rho \hat{\rho} (\Omega_0 - \hat{\Omega}_0)^2}{(\rho + \hat{\rho})^2} + \frac{\rho \omega_A^2 + \hat{\rho} \hat{\omega}_A^2}{\rho + \hat{\rho}} - \frac{k_0 (\rho - \hat{\rho}) g}{\rho + \hat{\rho}}}. \quad (38)$$

⇒ Stable (square root real) if

$$(\mathbf{k}_0 \cdot \mathbf{B})^2 + (\mathbf{k}_0 \cdot \hat{\mathbf{B}})^2 > \frac{\rho \hat{\rho}}{\rho + \hat{\rho}} [\mathbf{k}_0 \cdot (\mathbf{v} - \hat{\mathbf{v}})]^2 + k_0 (\rho - \hat{\rho}) g. \quad (39)$$

magnetic shear
K–H drive
R–T drive

Kelvin–Helmholtz instability: generic transitions

- **Pure KH instability** ($\mathbf{B} = \hat{\mathbf{B}} = 0, g = 0, \mathbf{k}_0 \parallel \mathbf{v} \parallel \hat{\mathbf{v}}$):

$$\omega = k_0 \left[\frac{\rho v + \hat{\rho} \hat{v}}{\rho + \hat{\rho}} \pm i \frac{\sqrt{\rho \hat{\rho}}}{\rho + \hat{\rho}} |v - \hat{v}| \right]. \quad (40)$$

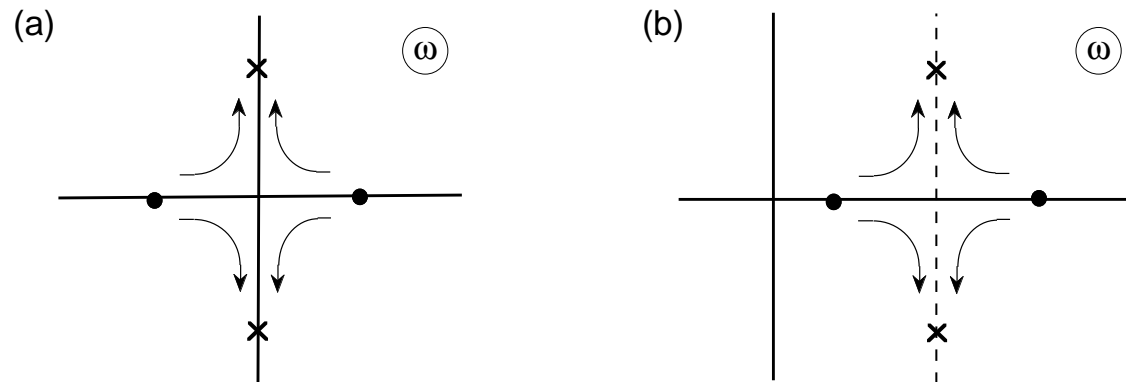
⇒ Degeneracy of Doppler mode $\omega = k_0 v$ lifted by $v \neq \hat{v}$.

- **Doppler shifted RT instability** ($\mathbf{B} = \hat{\mathbf{B}} = 0, \mathbf{v} = \hat{\mathbf{v}}, \mathbf{k}_0 \parallel \mathbf{v}$):

$$\omega = k_0 v \pm i \sqrt{\frac{k_0(\rho - \hat{\rho})g}{\rho + \hat{\rho}}}. \quad (41)$$

⇒ Degeneracy of Doppler mode $\omega = k_0 v$ lifted by $\rho \neq \hat{\rho}$.

- Hence, generic *transitions to instability* for (a) static, and (b) stationary plasmas:



Exp. growth: through origin

Overstability: through real axis

Kelvin–Helmholtz instability: generalizations

- Of course, the assumption of two homogeneous plasma layers with a velocity difference at the interface (made to make the analysis tractable for a relevant instability) evades the basic problems of **diffuse plasma flows**: continuous spectra, cluster points, and *eigenvalues on unknown paths in the complex ω plane*.
 - ⇒ Further progress only by *linear computational methods*: finite differences and finite elements, spectral methods, linear system solvers, etc.
- Instabilities always grow towards amplitudes that necessitate consideration of the **nonlinear evolution**: *coupling of linear modes, nonlinear saturation, and turbulence* appear: see simulation of Rayleigh–Taylor instability with Versatile Advection Code, where secondary Kelvin–Helmholtz instabilities develop (sheet 6-42).
 - ⇒ Further progress mainly by *nonlinear computational methods*: implicit and semi-implicit time stepping, finite volume methods, shock-capturing methods, etc.

Recap.: MHD wave equation in cylinder (static)

 [Vol. 1: Chap. 9]

- **Fourier harmonics** $\hat{\xi}(r; m, k) \exp [i(m\theta + kz)]$, keep differential operators d/dr .
- **Field line projection** in normal, perpendicular, and parallel directions:

$$\mathbf{e}_r \equiv \nabla r, \quad \mathbf{e}_\perp(r) \equiv (\mathbf{B}/B) \times \mathbf{e}_r, \quad \mathbf{e}_\parallel(r) \equiv \mathbf{B}/B, \quad (42)$$

$$\partial_r \equiv d/dr,$$

$$\nabla = \mathbf{e}_r \partial_r + \mathbf{e}_\theta r^{-1} \partial_\theta + \mathbf{e}_z \partial_z \Rightarrow G(r) \equiv mB_z/r - kB_\theta, \quad (43)$$

$$(\text{watch out: } \partial_\theta \mathbf{e}_r = \mathbf{e}_\theta, \quad \partial_\theta \mathbf{e}_\theta = -\mathbf{e}_r!) \quad F(r) \equiv mB_\theta/r + kB_z,$$

$$\xi = \xi \mathbf{e}_r - i\eta \mathbf{e}_\perp - i\zeta \mathbf{e}_\parallel. \quad (44)$$

MHD spectral equation $\mathbf{F}(\xi) = -\rho\omega^2 \xi$ (+ algebra!) \Rightarrow **Vector eigenvalue problem:**

$$\begin{pmatrix} \frac{d}{dr} \frac{\gamma p + B^2}{r} \frac{d}{dr} r - F^2 - r \left(\frac{B_\theta^2}{r^2} \right)' & \frac{d}{dr} \frac{G}{B} (\gamma p + B^2) - 2 \frac{kB_\theta B}{r} & \frac{d}{dr} \frac{F}{B} \gamma p \\ -\frac{G}{B} \frac{\gamma p + B^2}{r} \frac{d}{dr} r - 2 \frac{kB_\theta B}{r} & -\frac{G^2}{B^2} (\gamma p + B^2) - F^2 & -\frac{FG}{B^2} \gamma p \\ -\frac{F}{B} \frac{\gamma p}{r} \frac{d}{dr} r & -\frac{FG}{B^2} \gamma p & -\frac{F^2}{B^2} \gamma p \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = -\rho\omega^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \quad (45)$$

- Eliminate perpendicular and parallel components η and ζ :

$$\eta = \frac{\rho(\gamma p + B^2)G(\omega^2 - \omega_S^2) r\chi' + 2kB_\theta(B^2\rho\omega^2 - \gamma pF^2)\chi}{r^2BD},$$

$$\zeta = \frac{\rho\gamma pF[(\omega^2 - \omega_A^2) r\chi' + 2kB_\theta G\chi]}{r^2BD},$$
(46)

- Substitute in 1st component \Rightarrow **generalized Hain–Lüst equation:**

$$\frac{d}{dr} \left[\frac{N}{rD} \frac{d\chi}{dr} \right] + \frac{1}{r} \left[\rho\omega^2 - F^2 - r \left(\frac{B_\theta^2}{r^2} \right)' - \frac{4k^2 B_\theta^2}{r^2 D} (B^2\rho\omega^2 - \gamma pF^2) + r \left\{ \frac{2kB_\theta G}{r^2 D} ((\gamma p + B^2)\rho\omega^2 - \gamma pF^2) \right\}' \right] \chi = 0, \quad (47)$$

with singular factors

$$N = N(r; \omega^2) \equiv \rho^2(\gamma p + B^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_S^2),$$

$$D = D(r; \omega^2) \equiv \rho^2\omega^4 - (m^2/r^2 + k^2)(\gamma p + B^2)(\omega^2 - \omega_S^2).$$
(48)

- BCs:

$$\chi(0) = \chi(a) = 0 \quad (\text{including } m = 1 \text{ at } r = 0). \quad (49)$$

MHD wave equation in cylinder with flow

- Generalizing the MHD wave equation to equilibria with background flow, exploiting the MHD spectral equation $\mathbf{G}(\boldsymbol{\xi}) + 2i\rho\omega\mathbf{v} \cdot \nabla\boldsymbol{\xi} + \rho\omega^2\boldsymbol{\xi} = 0$ (+ a lot of algebra!)
 \Rightarrow **Quadratic vector eigenvalue problem:**

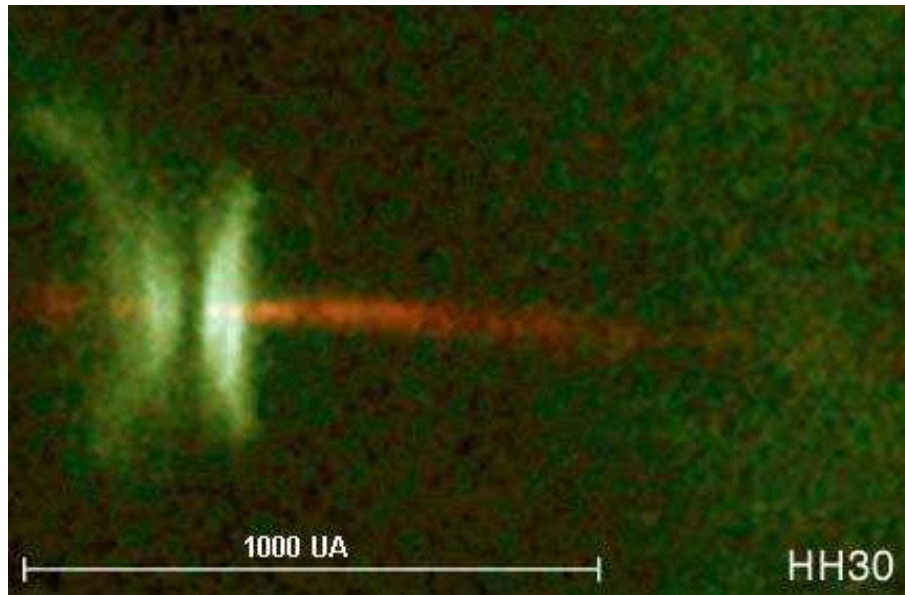
$$\left[\mathbf{F}_0 + \begin{pmatrix} -r\rho\left(\frac{\Phi'_{\text{gr}}}{r}\right)' & -\Lambda\frac{G}{B} & -\Lambda\frac{F}{B} \\ -\Lambda\frac{G}{B} & 0 & 0 \\ -\Lambda\frac{F}{B} & 0 & 0 \end{pmatrix} - 2\rho\frac{v_\theta}{r}\tilde{\omega} \begin{pmatrix} 0 & \frac{B_z}{B} & \frac{B_\theta}{B} \\ \frac{B_\theta}{B_z} & 0 & 0 \\ \frac{B_\theta}{B_\theta} & 0 & 0 \end{pmatrix} + \rho\tilde{\omega}^2 \mathbf{I} \right] \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = 0, \quad (50)$$

where \mathbf{F}_0 is the matrix on the LHS of the static wave equation (45), the function $\Lambda(r) \equiv \rho(v_\theta^2/r - \Phi'_{\text{gr}})$ represents the deviation from static equilibrium due to rotation and gravity (or the deviation of HD equilibria from Keplerian flow), the Doppler shifted frequency $\tilde{\omega}(r) \equiv \omega - \Omega_0(r)$, where $\Omega_0 \equiv mv_\theta/r + kv_z$, and \mathbf{I} is the unit matrix.

- Elimination of η and ζ and substitution in the first component, yields again a 2nd order ODE like the Hain–Lüst equation: see Eq. (51) below.

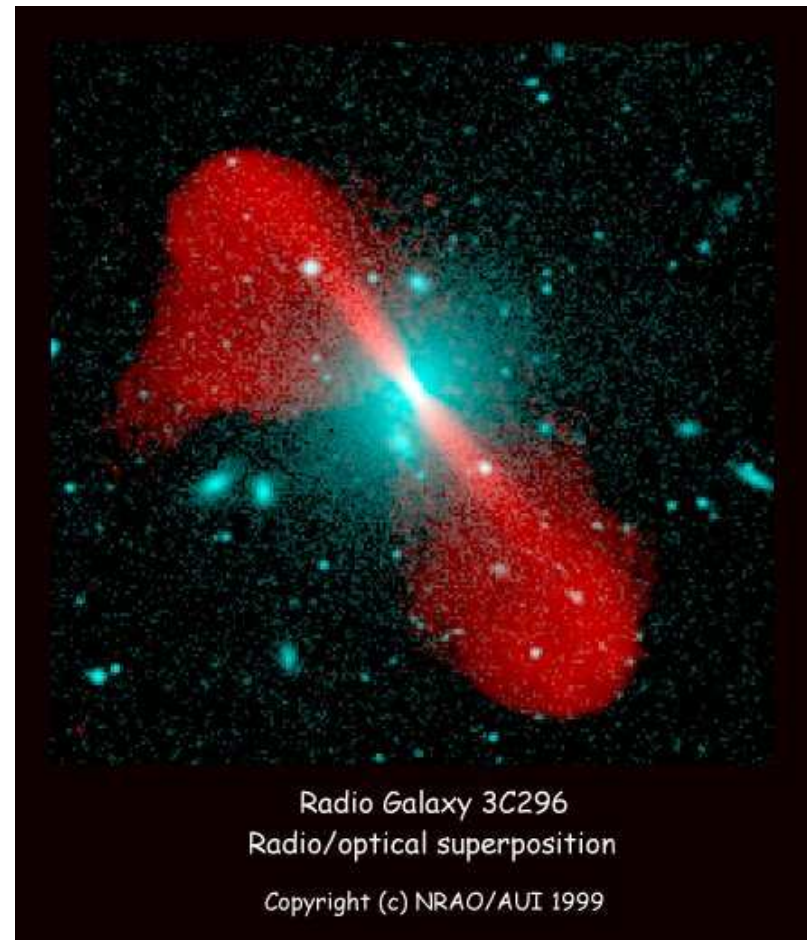
Observations

Young Stellar Object ($M_* \sim 1M_\odot$)



disk and jets

Active Galactic Nucleus ($\sim 10^8 M_\odot$)



disk (optical) and jets (radio)

Magneto-rotational instability

- Example of **cylindrical flow**. Original references:
 - Velikhov, Soviet Phys.–JETP Lett. **36**, 995 (1959);
 - Chandrasekhar, Proc. Nat. Acad. Sci. USA **46**, 253 (1960).
- Applied to *accretion disks* by Balbus and Hawley, Astrophys. J. **376**, 214 (1991).
Problem: how can accretion on Young Stellar Object (mass $M_* \sim M_\odot$) or Active Galactic nucleus (mass $M_* \sim 10^9 M_\odot$) occur at all on a reasonable time scale?
 - Without dissipation impossible, because disk would conserve angular momentum; some form of viscosity needed to transfer angular momentum to larger distances.
 - However, ordinary molecular viscosity much too small to produce sizeable transfer, and for turbulent increase (small-scale instabilities) no HD candidates were found.
 - It is generally assumed that the resolution of this problem involves MHD instability: **the magneto-rotational instability (MRI)**.
- Simplify the axi-symmetric (2D) representation of the disk (see sheet 4-9) even further by *neglecting vertical variations* so that a **cylindrical (1D) slice** is obtained.
[One should object: but that is no disk at all anymore! Yet, this is how plasma-astrophysicists grapple with the problem of anomalous (turbulent) transport.]

MRI: cylindrical representation

- Generalization of Hain–Lüst equation, [Book, Eq. \(9.31\)](#), to **cylindrical flow with normal modes**

$$\xi \sim \exp [i(m\theta + kz - \omega t)],$$

again yields second order ODE for radial component of the plasma displacement:

$$\frac{d}{dr} \left[\frac{N}{rD} \frac{d\chi}{dr} \right] + \left[U + \frac{V}{D} + \left(\frac{W}{D} \right)' \right] \chi = 0, \quad \chi \equiv r\xi. \quad (51)$$

[Bondeson, Iacono and Bhattacharjee, Phys. Fluids **30**, 2167 (1987);

extended with gravity: Keppens, Casse, Goedbloed, Astrophys. J. **579**, L121 (2002)]

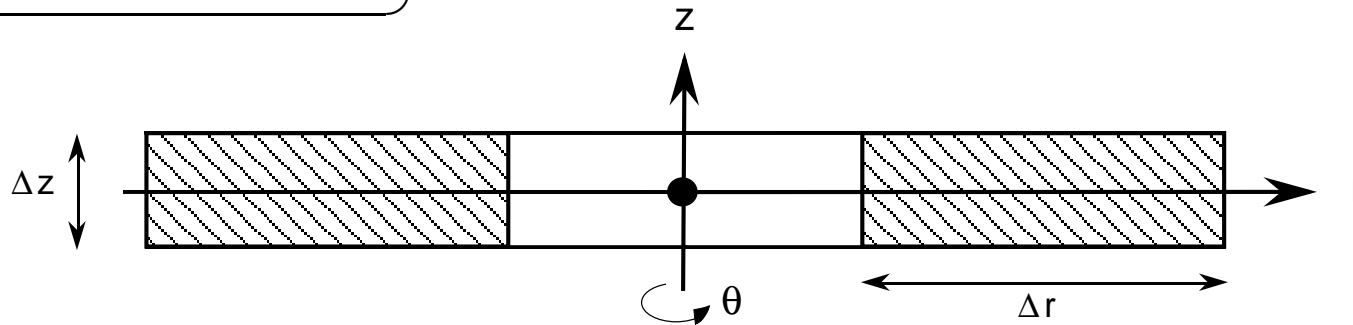
- Assumption of **small magnetic field**,

$$\beta \equiv 2p/B^2 \gg 1, \quad (52)$$

justifies use of this spectral equation in the **incompressible limit**:

$$\frac{d}{dr} \left[\frac{\rho\tilde{\omega}^2 - F^2}{m^2 + k^2r^2} r \frac{d\chi}{dr} \right] - \frac{1}{r} \left[\rho\tilde{\omega}^2 - F^2 + r \left(\frac{B_\theta^2 - \rho v_\theta^2}{r^2} \right)' + \rho' \Phi'_{\text{gr}} \right. \\ \left. - \frac{4k^2(B_\theta F + \rho v_\theta \tilde{\omega})^2}{(m^2 + k^2r^2)(\rho\tilde{\omega}^2 - F^2)} - r \left(\frac{2m(B_\theta F + \rho v_\theta \tilde{\omega})}{r(m^2 + k^2r^2)} \right)' \right] \chi = 0. \quad (53)$$

MRI: approximations



- Gravitational potential of compact object is approximated for cylindrical slice,

$$\Phi_{\text{gr}} = -GM_*/\sqrt{r^2 + z^2} \approx -GM_*/r, \quad (54)$$

with *short wavelengths fitting the disk in the vertical direction:*

$$k \Delta z \gg 1. \quad (55)$$

- Incompressibility is consistent with *constant density* so that the only gravitational term, $\rho' \Phi'_{\text{gr}}/r$, disappears from the spectral equation. However, Φ_{gr} does not disappear from the equilibrium equation that ρ , p , B_θ , B_z , and v_θ have to satisfy,

$$(p + \frac{1}{2}B^2)' + \frac{1}{r}B_\theta^2 = \frac{1}{r}\rho v_\theta^2 - \rho \Phi'_{\text{gr}},$$

so that *stability will still be influenced by gravity.*

MRI: further approximations

- Assume *purely vertical and constant magnetic field* and *purely azimuthal velocity*,

$$B_\theta = 0, \quad v_z = 0 \quad \Rightarrow \quad \omega_A = kB_z/\sqrt{\rho} = \text{const}, \quad \Omega_0 = mv_\theta/r, \quad (56)$$

and restrict analysis to *vertical wave numbers* k only,

$$m = 0 \quad \Rightarrow \quad \Omega_0 = 0 \quad \Rightarrow \quad \tilde{\omega} = \omega. \quad (57)$$

In the spectral equation, only ω^2 appears: $\omega = 0$ **remains the marginal point!**

$$(\omega^2 - \omega_A^2) \frac{d}{dr} \left(\frac{1}{r} \frac{d\chi}{dr} \right) - \frac{k^2}{r} \left[\omega^2 - \omega_A^2 - r \left(\frac{v_\theta^2}{r^2} \right)' - \frac{4\omega^2 v_\theta^2 / r^2}{\omega^2 - \omega_A^2} \right] \chi = 0. \quad (58)$$

- Introduce *angular frequency* $\Omega \equiv v_\theta/r$ and *epicyclic frequency* κ ,

$$\kappa^2 \equiv \frac{1}{r^3} (r^4 \Omega^2)' = 2r\Omega\Omega' + 4\Omega^2 \quad (59)$$

(Specific angular momentum $L \equiv \rho r v_\theta \equiv \rho r^2 \Omega$; hence $\kappa^2 = 0 \Rightarrow L' = 0$.)

The spectral equation then becomes:

$$(\omega^2 - \omega_A^2) \frac{d}{dr} \left(\frac{1}{r} \frac{d\chi}{dr} \right) - \frac{k^2}{r} \left[\omega^2 - \omega_A^2 - \kappa^2(r) - \frac{4\omega_A^2 \Omega^2(r)}{\omega^2 - \omega_A^2} \right] \chi = 0. \quad (60)$$

MRI: criteria

- Recall construction of quadratic form (sheet 7-24e):

$$(P\chi')' - Q\chi = 0 \quad \Rightarrow \quad \int (P\chi'^2 + Q\chi^2) r dr = 0. \quad (61)$$

\Rightarrow For eigenfunctions (oscillatory χ), we should have $Q/P < 0$ for some r .

- From Eq. (60), this gives the following *criteria for instability* ($\omega^2 < 0$):

$$\begin{aligned} \text{(a) MHD } (\omega_A^2 \neq 0): \quad & \omega_A^2 + \kappa^2 - 4\Omega^2 < 0 \\ & \text{(for some range of } r \text{).} \quad (62) \\ \text{(b) HD } (\omega_A^2 \equiv 0): \quad & \kappa^2 < 0 \end{aligned}$$

- For *Keplerian rotation* (neglecting p and B on equilibrium motion):

$$\frac{1}{r}\rho v_\theta^2 = \rho\Phi'_{\text{gr}} = \rho\frac{GM_*}{r^2} \quad \Rightarrow \quad \Omega^2 = \frac{GM_*}{r^3} \quad \Rightarrow \quad \kappa^2 = \frac{GM_*}{r^3} > 0. \quad (63)$$

\Rightarrow In HD limit, opposite of (62)(b) holds, *Rayleigh's circulation criterion is satisfied: the fluid is stable* to axi-symmetric modes ($m = 0$) if $\kappa^2 \geq 0$ everywhere.

This explains the interest in MHD instabilities as candidates for turbulent increase of the dissipation processes in accretion disks.

MRI: MHD versus HD

- **MHD instability criterion in the limit $\omega_A^2 \rightarrow 0$** (magnetic field sufficiently small):

$$\kappa^2 - 4\Omega^2 \equiv 2r\Omega\Omega' < 0. \quad (64)$$

This is **satisfied for Keplerian disks**: MRI works for astrophysically relevant cases! Stabilizing field contribution ($\omega_A^2 > 0$) should be small enough to maintain this effect.

- Discrepancy of HD and MHD stability results is due to *interchange of limits*:

$$\text{HD disk: } \omega_A^2 = 0, \omega^2 \rightarrow 0, \quad \text{MHD disk: } \omega^2 = 0, \omega_A^2 \rightarrow 0.$$

This is resolved when the **growth rates** of the instabilities are considered.

- Instead of numerically solving ODE (60), just consider radially localized modes, $\chi \sim \exp(iqr)$, $q\Delta r \gg 1$, producing a *local dispersion equation*:

$$(k^2 + q^2)(\omega^2 - \omega_A^2)^2 - k^2\kappa^2(\omega^2 - \omega_A^2) - 4k^2\omega_A^2\Omega^2 = 0. \quad (65)$$

Solutions for $q^2 \ll k^2$:

$$\omega^2 = \omega_A^2 + \frac{1}{2}\kappa^2 \pm \frac{1}{2}\sqrt{\kappa^4 + 16\omega_A^2\Omega^2} \approx \begin{cases} \kappa^2 + \omega_A^2(1 + 4\Omega^2/\kappa^2) \\ \omega_A^2(1 - 4\Omega^2/\kappa^2) \end{cases}, \quad (66)$$

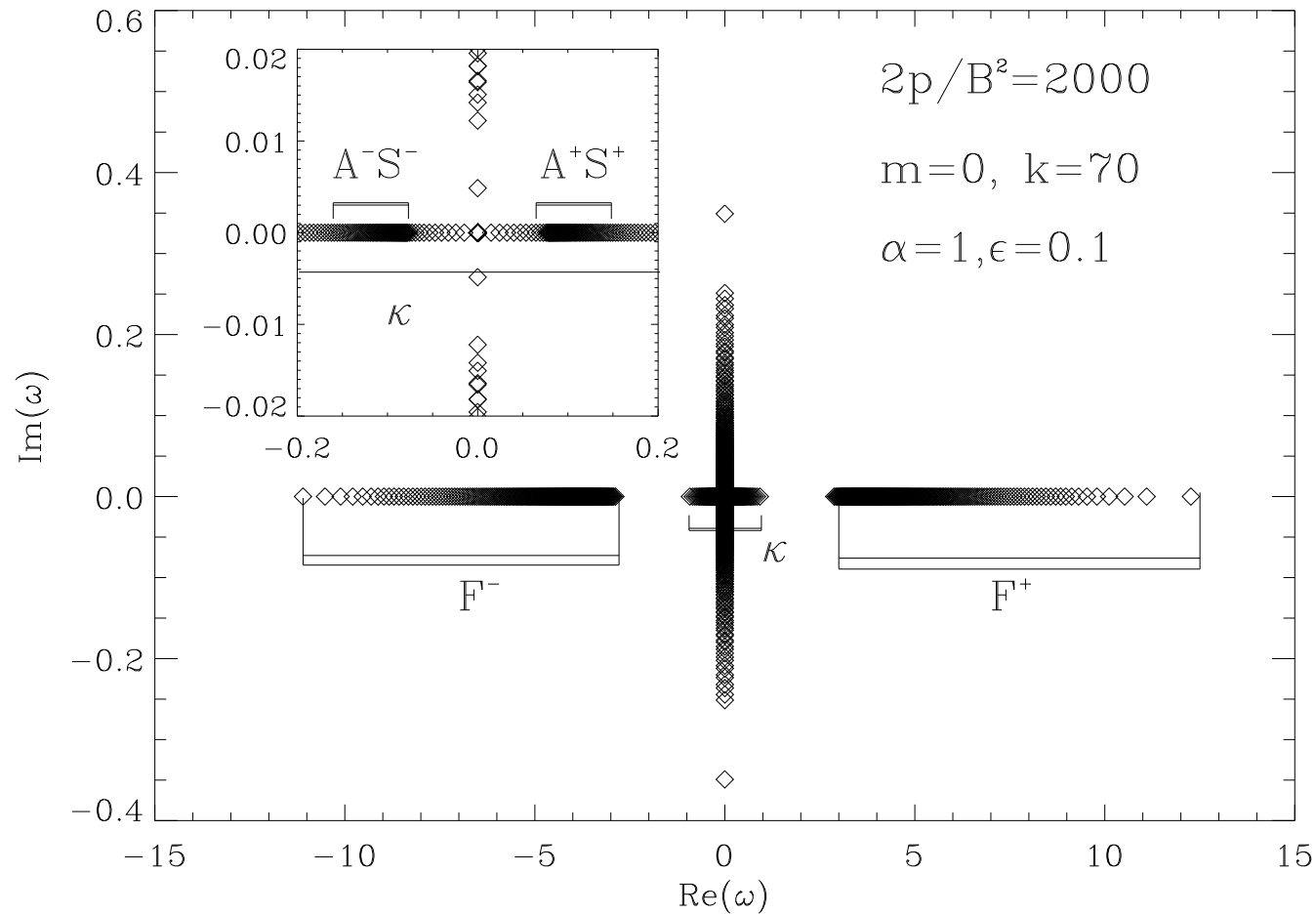
Limit $\omega_A^2 \rightarrow 0$: (1) **Rayleigh mode (HD)**, $\omega_+^2 \rightarrow \kappa^2 > 0$, (2) **MRI (MHD)**, $\omega_-^2 \rightarrow 0$.

MRI: Numerical results

[Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]

- **Full spectrum with toroidal field ($m = 0$)**

⇒ Discrete MRIs form cluster spectrum towards slow continuum:



MRI: Numerical results

[Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]

- **Full spectrum with toroidal field ($m = 10$)**

⇒ Many more unstable sequences in the complex ω -plane:

