Explicit approximations to estimate the perturbative diffusivity in the presence of convectivity and damping (Part 2): Semi-infinite cylindrical approximations

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Abstract

In this paper, a number of new explicit approximations are introduced to estimate the perturbative diffusivity (χ), convectivity (V), and damping (τ) in a cylindrical geometry. For this purpose the harmonic components of heat waves induced by localized deposition of modulated power are used. The approximations are based upon the heat equation in a semi-infinite cylindrical domain. The approximations are based upon continued fractions, asymptotic expansions, and multiple harmonics.

The relative error for the different derived approximations is presented for different values of frequency, transport coefficients, and dimensionless radius. Moreover, it is shown how combinations of different explicit formulas can yield good approximations over a wide parameter space for different cases, such as no convection and damping, only damping, and both convection and damping.

This paper is the second part (Part 2) of a series of three papers. In Part 1 the semi-infinite slab approximations have been treated. In Part 3 cylindrical approximations are treated for heat waves traveling towards the center of the plasma.

I.   INTRODUCTION

This paper, Part 2 of a series of three papers, deals with semi-infinite cylindrical approximations that can be used to calculate the perturbative thermal diffusion coefficient, convectivity, and damping from the experimental data of heat pulse propagation. For a general introduction of the series of three papers the reader is referred to [1].

This paper is structured as follows. Section II gives an overview of the relevant assumptions and models used for perturbative transport analysis. In Section III, three methods are introduced to find approximations for χ, V, and τ, i.e., continued fractions, asymptotic expansions, and multiple harmonics. Section IV gives an overview of possible explicit approximations under various assumptions. The different approximations are compared for different values of the transport coefficients, dimensionless radius, and frequency. In practice, the transport coefficients are unknown. Therefore, in Section V a method is introduced to select and test the accuracy of the approximation without using knowledge of the actual transport coefficients. Finally, in Section VI the main results are summarized and discussed.

II.   MODELING OF THERMAL TRANSPORT

In this section, the main assumptions and models used for perturbative transport analysis of the electron transport are summarized.

A.   Perturbative transport analysis

Linearized thermal transport inside a fusion reactor is often modeled as radial (1D) transport in a cylinder due to the magnetic confined plasma topology [1–3]

$$\frac{3}{2}\frac{\partial}{\partial t}\left(nT\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(n\rho\chi\left(\rho\right)\frac{\partial T}{\partial \rho} + n\rho V\left(\rho\right)T\right)$$

$$-\frac{3}{2}n\tau_{inv}\left(\rho\right)T + P_{mod}, \tag{1}$$

where χ is the diffusivity, V the convectivity, τ_{inv} the (inverse) damping ($\tau_{inv} = 1/\tau$), T denotes the electron temperature, n the density, ρ the dimensionless radius, and P_{mod} a perturbative heat source. However, a number of assumptions are necessary to derive direct expressions for χ, which are standard in the literature [2, 4]. These assumptions simplify

(1) such that analytic solutions can be derived for (1). Only measurements are considered for which the transients due to the initial condition can be neglected. It is assumed that the parameters are constant with respect to time and ρ, which is also referred to as being homogenous or uniform [4]. Finally, only spatial regions are considered where $P_{mod} = 0$, i.e., outside the region where the heating is deposited to perturb the plasma. Here, it is additionally assumed that the density n is constant with respect to ρ. Under these assumptions (1) can be transformed into the Laplace domain yielding

$$\frac{3}{2}(s + \tau_{inv}) \Theta(\rho, s) =$$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \chi \frac{\partial \Theta(\rho, s)}{\partial \rho} + \rho V \Theta(\rho, s) \right), \tag{2}$$

where Θ is the Laplace transformed temperature T and s is the Laplace variable. This complex valued Ordinary Differential Equation (ODE) can be solved analytically [5, 6]:

$$\Theta(\rho, s) = e^{\lambda_1 \rho} D_1(s) \Psi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, (\lambda_2 - \lambda_1)\rho\right)$$

$$+ e^{\lambda_1 \rho} D_2(s) \Phi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, (\lambda_2 - \lambda_1)\rho\right), \tag{3}$$

where

$$\lambda_{1,2} = -\frac{V}{2\chi} \mp \sqrt{\left(\frac{V}{2\chi}\right)^2 + \frac{3}{2}\frac{s + \tau_{inv}}{\chi}}. \tag{4}$$

The functions $\Phi(\rho, s)$ and $\Psi(\rho, s)$ denote the Confluent Hypergeometric Functions of the first and the second kind, respectively. These functions are also often denoted as $_1F_1$ and U, and are extensively described in [7–9]. The boundary constants are denoted by $D_1(s)$ and $D_2(s)$. This solution in terms of Confluent Hypergeometric Functions may not be so familiar, but if simplified by assuming $V = 0$, the well-known solutions in terms of the modified Bessel functions of the first kind I_ν and the second kind K_ν of order $\nu = 0$ are found. In particular, if $V = 0$, then $\lambda_1 = -\lambda_2$, $\lambda_2/(\lambda_2 - \lambda_1) = 1/2$, and $\Phi(\rho, s)$ and $\Psi(\rho, s)$ result in [7]

$$\Phi\left(\frac{1}{2}, 1, (\lambda_2 - \lambda_1)\rho\right) =$$

$$\exp\left(\frac{1}{2}(\lambda_2 - \lambda_1)\rho\right) I_0\left(\frac{1}{2}(\lambda_2 - \lambda_1)\rho\right) \tag{5}$$

and

$$\Psi\left(\frac{1}{2}, 1, (\lambda_2 - \lambda_1)\rho\right) =$$

$$\exp\left(\frac{1}{2}\left(\lambda_2 - \lambda_1\right)\rho\right)\frac{K_0\left(\frac{1}{2}\left(\lambda_2 - \lambda_1\right)\rho\right)}{\sqrt{\pi}}. \tag{6}$$

So (3) simplifies to

$$\Theta\left(\rho, s\right) = \frac{D_1\left(s\right)}{\sqrt{\pi}}K_0\left(\frac{1}{2}\left(\lambda_2 - \lambda_1\right)\rho\right)$$

$$+ D_2\left(s\right)I_0\left(\frac{1}{2}\left(\lambda_2 - \lambda_1\right)\rho\right). \tag{7}$$

This solution is well known and is studied and referenced in e.g. [6, 10].

These solutions are used to derive hands-on models, which can be used to derive explicit approximations of χ.

B.   Logarithmic temperature derivative and transfer function

An explicit solution is difficult to derive on the basis of the two eigenfunctions $\Phi\left(\rho, s\right)$ and $\Psi\left(\rho, s\right)$. Therefore, an additional simplification step is necessary to be able to eliminate one eigenfunction. One eigenfunction is eliminated by assuming a semi-infinite domain, which is defined as follows, if $\rho \to \infty$, then $\Theta \to 0$. This means that at $\rho = \infty$ all perturbations need to have vanished.

In case $V = 0$ then for (7) it means that $D_2\left(s\right) = 0$, which is well known in the literature [10]. This also implies that $D_2\left(s\right) = 0$ in (3). However, this is difficult to prove analytically. Instead, it is numerically verified that $D_2\left(s\right) = 0$ in (3) by comparing it to finite difference simulations with boundary condition $T\left(\rho \gg 1\right) = 0$. This shows that the error between the analytic and numerical simulations are small and that the error is decreasing with increasing density of the discretization grid. In addition, the functions Φ and Ψ are numerically evaluated to study the behavior close to $\rho = \infty$, indicating again that this is the correct choice. Hence, it is concluded that $D_2\left(s\right) = 0$ for a semi-infinite domain such that (3) becomes

$$\Theta\left(\rho, s\right) = D_1\left(s\right)e^{\lambda_1\rho}\Psi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, \left(\lambda_2 - \lambda_1\right)\rho\right). \tag{8}$$

In principle, $D_1\left(s\right)$ needs to be fixed by assuming a second boundary condition. However, it is common practice to express the solution in terms of the spatial logarithmic derivative of the amplitude A and the spatial derivative of ϕ, which can be calculated using the logarithmic temperature derivative $\left(\partial\Theta/\partial\rho\right)/\Theta$. Therefore, the temperature Θ is written in terms of its harmonic components, i.e., $\Theta = A\exp\left(i\phi\right)$, and the spatial derivative of the temperature is

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given by $\Theta' = A' \exp(i\phi) + i\phi' A \exp(i\phi)$ such that

$$\frac{\Theta'}{\Theta} = \frac{A'}{A} + i\phi'. \tag{9}$$

Taking the logarithmic derivative of the right hand side of (8) results in, see [8]

$$\frac{\Theta'}{\Theta} = \lambda_1 - \lambda_2 \frac{\Psi\left(1 + \frac{\lambda_2}{\lambda_2 - \lambda_1}, 2, (\lambda_2 - \lambda_1)\rho\right)}{\Psi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, (\lambda_2 - \lambda_1)\rho\right)}, \tag{10}$$

which will be used to derive an explicit relationship to identify χ in the presence of V and τ_{inv}. If $V = 0$, then (10) simplifies to

$$\frac{\Theta'}{\Theta} = -z\frac{K_1(z\rho)}{K_0(z\rho)} \tag{11}$$

with $z = \frac{1}{2}(\lambda_2 - \lambda_1)$. This relationship is well known in the literature [10].

A common problem using the representations (10) and (11) is how to calculate the spatial derivatives A'/A and ϕ' from the measured A and ϕ. In contrast to the slab-geometry case, where the spatial derivative can be explicitly calculated, see [1], for cylindrical geometry the spatial derivative cannot be easily calculated. On the other hand, it is possible to avoid the calculation of the spatial derivatives by directly expressing the transport coefficients and ρ in terms of the measured amplitudes using the transfer function description [1, 11–13]. This description needs a second boundary condition to define $D_1(s)$, which basically defines the relationship between A'/A and A, and ϕ' and ϕ.

The most logical choice for a second boundary condition is $\Theta(\rho, s) = \Theta(\rho_1, s)$, because $\Theta(\rho_1, s)$ is measured. In addition, the domain on which the transport coefficients are analyzed cannot contain a source term, but still this domain should be limited by a boundary condition. Using the same steps as presented in [1], the transfer function from the temperature at point ρ_1 to the temperature at point ρ_2 is given by

$$\frac{\Theta(\rho_2, s)}{\Theta(\rho_1, s)} = e^{\lambda_1 \Delta\rho} \frac{\Psi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, (\lambda_2 - \lambda_1)\rho_2\right)}{\Psi\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}, 1, (\lambda_2 - \lambda_1)\rho_1\right)}, \tag{12}$$

where the solution at a second measurement point $\rho_2 > \rho_1$ is used as resulting temperature $\Theta(\rho_2)$. The transfer function description avoids the ambiguity of how to determine A'/A and ϕ'. On the other hand, from the transfer function description in cylindrical geometry it is not straightforward to derive explicit relationships for χ. On the other hand, the transfer

description can be used to validate the estimated coefficients by comparing the measured left hand side to the right hand side where the estimated χ, V, and τ_{inv} need to be substituted. In particular, as the transfer function can be derived for many different choices of boundary conditions, e.g., a Dirichlet boundary condition or a Robin boundary condition at the edge could be used. As such, it is possible to check if the semi-infinite domain assumption is valid. However, in this paper this will not be further investigated. In the next section, approximations are derived to estimate χ, V, and τ_{inv}.

III.   DERIVATION OF EXPLICIT APPROXIMATIONS

Here, three new methods are introduced to find approximations for determining the diffusivity in cylindrical geometry using (10) and (11). They are based on continued fractions, asymptotic expansions, and multiple harmonics. A great number of new approximations are derived in this paper. Therefore, in this section only the basic ideas are explained to increase the readability of the paper. The full derivations are summarized in the next section and their derivations can be found in the appendix.

A.   Continued fractions

The solution to the Bessel Differential Equation and Confluent Hypergeometric Differential Equation can be found by substitution of a power series. This yields the recurrence relationship, which is used to calculate the infinite series representation of Bessel functions and Confluent Hypergeometric Functions. Moreover, if a fraction of these functions is studied the recurrence relationships and series representation can also be rewritten in terms of a Continued Fractions representation [14, 15]. By truncating this Continued Fraction the ratio of Bessel functions and Confluent Hypergeometric Functions can be approximated. Here, this approximation is inverted such that the transport coefficients can be directly calculated.

The use of a continued fraction to derive an explicit approximation for χ is best explained by means of an example. Therefore, consider (11) where for $V = 0$ such that a new variable z can be defined:

$$z = \sqrt{\frac{3}{2}\frac{i\omega + \tau_{inv}}{\chi}}. \tag{13}$$

The Laplace variable is replaced by $s = i\omega$ as s can only be measured on the imaginary axis. The Continued J-fraction from [14, 15] can be used to calculate the ratio

$$\frac{K_1(z\rho)}{K_0(z\rho)} = 1 + \frac{1}{2z\rho}$$

$$- \cfrac{(4z\rho)^{-1}}{2z\rho + 2 - \cfrac{9/4}{2z\rho + 4 - \cfrac{25/4}{2z\rho + 6 - \cfrac{64/4}{\cdots}}}}. \tag{14}$$

This continued fraction can be truncated to find explicit approximations for χ. The most simple truncation is

$$\frac{\Theta'}{\Theta} = -z\left(1 + \frac{1}{2z\rho}\right). \tag{15}$$

This can be rewritten by substituting (9) and rewriting in terms of z

$$z = -\left(\frac{A'}{A} + i\phi' + \frac{1}{2\rho}\right). \tag{16}$$

Now by substituting (13) and squaring (16) it is possible to find χ and τ_{inv} using the imaginary part and real part

$$\chi_c = \frac{3}{4}\frac{\omega}{\left(\frac{A'}{A} + \frac{1}{2\rho}\right)\phi'} \tag{17}$$

and

$$\tau_c = \frac{\omega}{2}\left(\frac{\left(\frac{A'}{A} + \frac{1}{2\rho}\right)}{\phi'} - \frac{\phi'}{\left(\frac{A'}{A} + \frac{1}{2\rho}\right)}\right). \tag{18}$$

The index, here c, is used to distinguish between the different approximations. In [4] the same approximation is found based on a different method. However, unlike [4] using this method also the corresponding formula to calculate τ_{inv} is found.

This method can also be used to find more accurate approximations by using more terms in the continued fraction.

In principle, it is possible to find an approximation of arbitrary accuracy. However, in practice solving for χ beyond third order polynomials in z is too complicated. The reason is that finding zeros for a fourth order polynomial in z is not straightforward. In addition, different solutions apply for different regions of interest, because the coefficients in the polynomials in z depend on A'/A and ϕ'. Hence, mathematical bifurcations can occur.

The continued fraction in (14) can also be truncated at a later stage, for the first three terms of (14) this results in

$$\frac{\Theta'}{\Theta} = -z + \frac{1}{2\rho} + \frac{1}{\rho}\left(-\cfrac{1/4}{2z\rho + 2 - \cfrac{9/4}{2z\rho + 4}}\right). \tag{19}$$

It can be rewritten in terms of z resulting in the third order polynomial

$$0 = a_3 z^3 + a_2 z^2 + a_1 z + a_0, \tag{20}$$

with coefficients

$$a_3 = 16\rho^3, a_2 = 16\frac{\Theta'}{\Theta}\rho^3 + 56\rho^2,$$

$$a_1 = 48\frac{\Theta'}{\Theta}\rho^2 + 45\rho, \text{ and } a_0 = 23\frac{\Theta'}{\Theta}\rho + 7.5, \tag{21}$$

where Θ'/Θ is given by (9). The third order polynomial yields three solutions. However, generally only one solution can be used to determine χ, because the argument of z is situated in the domain $[0, \pi/4]$ (χ, τ_{inv}, and ω are larger than zero) and the other two are outside this domain. This has been numerically calculated and it turns out that the correct zero is given by

$$z = -\frac{a_2}{3a_3} - \frac{\sqrt[3]{2}p_0}{3a_3 p_1} + \frac{p_1}{3\sqrt[3]{2}a_3}, \tag{22}$$

where $p_0 = 3a_1 a_3 - a_2^2$, $p_1 = \sqrt[3]{p_2 + \sqrt{4p_0^3 + p_2^2}}$, and $p_2 = -27a_0 a_3^2 + 9a_1 a_2 a_3 - 2a_2^3$. Now the solution is found by squaring and using the imaginary part and the real part of z^2

$$\chi = \frac{3}{2}\frac{\omega}{\Im(z^2)}. \tag{23}$$

The solution for the semi-infinite domain approximations of a third order polynomial, i.e., (20), is always given by (22). This is because the argument of z of the correct solution must be situated in $[0, \pi/4]$. This is difficult to proof, but has been numerically verified. This holds similarly for a second order polynomial in z. Hence, χ can always be found for such polynomials in z using (23), which is based on (13). Therefore, in the appendix only the polynomials in terms of z are given. The approximation for τ_{inv} can also be found based on (13)

$$\tau_{inv} = \omega\frac{\Re(z^2)}{\Im(z^2)}. \tag{24}$$

However, the damping τ_{inv} is much more sensitive to errors due to mismodeling [1]. Therefore, they generally do not give reasonable values for the true damping. This holds also for approximations of the convectivity V, which are derived in the next sections. Nevertheless, they play an important role in the validation of the χ estimates. In the appendix, other continued fractions are used to find explicit approximations for χ.

B. Asymptotic expansions

An alternative to continued fractions are asymptotic expansions. An asymptotic expansion of a function can be based on either a truncation of its series expansion or on an approximation of the defining Ordinary Differential Equation for the function [7, 8, 16]. The asymptotic expansions can be calculated for the Bessel functions and for the Confluent Hypergeometric Function of the second kind Ψ. Here, an approximation is derived based on (10) in terms of Ψ and in the appendix the derivation of the approximation for χ based on Bessel functions of the second kind can be found.

In [8, 17] the asymptotic expansion for $x \approx 0$ of $\Psi(a, b, x)$ are given. Using this we find for $(\lambda_1 - \lambda_2)\rho$ small

$$\Psi\left(a, 1, (\lambda_2 - \lambda_1)\rho\right) =$$

$$-\frac{1}{\Gamma(a)}\left(\ln\left((\lambda_2 - \lambda_1)\rho\right) + F(a) + 2\gamma\right), \tag{25}$$

and

$$\Psi\left(a + 1, 2, (\lambda_2 - \lambda_1)\rho\right) = \frac{\left((\lambda_2 - \lambda_1)\rho\right)^{-1}}{\Gamma(a+1)}, \tag{26}$$

where $a = \lambda_2/(\lambda_2 - \lambda_1)$, γ is the Euler-Mascheroni constant and Γ is the gamma function. It should not be confused with the particle flux Γ, which is not considered in this paper. The logarithmic derivative of Γ is the digamma function denoted by F (often also denoted as ψ). The logarithmic temperature derivative can be calculated by substituting (25) and (26) into (10)

$$\frac{\Theta'}{\Theta} = \lambda_1 + \lambda_2 \frac{\frac{1}{\Gamma(a+1)}\left((\lambda_2 - \lambda_1)\rho\right)^{-1}}{\frac{1}{\Gamma(a)}\left(\ln\left((\lambda_2 - \lambda_1)\rho\right) + F(a) + 2\gamma\right)}. \tag{27}$$

This can be simplified using the property $\Gamma(a+1) = a\Gamma(a)$ and $a = \lambda_1/(\lambda_2 - \lambda_1)$

$$\frac{\Theta'}{\Theta} = \lambda_1 + \frac{1}{\rho\left(\ln\left((\lambda_2 - \lambda_1)\rho\right) + F(a) + 2\gamma\right)}. \tag{28}$$

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The digamma function $F(a)$ as function of a and the appearance of λ_1 in and outside the logarithm pose important obstacles to find approximations for χ. Therefore, two simplification steps are necessary. The asymptotic expansion is based on $x \approx 0$, which implies that $\rho \approx 0$. Thus, the term with $1/\rho$ will dominate over λ_1, hence (28) can be simplified to

$$\frac{\Theta'}{\Theta} = \frac{1}{\rho\left(\ln\left(\left(\lambda_2 - \lambda_1\right)\rho\right) + F(a) + 2\gamma\right)}. \tag{29}$$

However, if $V \neq 0$ the problem of $F(a)$ remains.

In case $V = 0$, $a = \frac{1}{2}$ such that $F\left(\frac{1}{2}\right) = -2\ln(2) - \gamma$ resulting in

$$\frac{\Theta'}{\Theta} = \frac{1}{\rho\left(\ln\left(\left(\lambda_2 - \lambda_1\right)\rho\right) - 2\ln(2) + \gamma\right)}, \tag{30}$$

rearranging yields

$$\lambda_2 - \lambda_1 = \frac{1}{\rho}\exp\left(\frac{1}{\frac{\Theta'}{\Theta}\rho} + 2\ln(2) - \gamma\right), \tag{31}$$

such that

$$\frac{1}{2}\left(\lambda_2 - \lambda_1\right) = \frac{\gamma_1}{\rho}\exp\left(\left(\frac{\Theta'}{\Theta}\rho\right)^{-1}\right), \tag{32}$$

where $\gamma_1 = 2\exp(-\gamma)$, which should be used with high accuracy in the calculations. The approximation in terms of z is given by

$$z = \frac{\gamma_1}{\rho}\exp\left(\left(\frac{\Theta'}{\Theta}\rho\right)^{-1}\right), \tag{33}$$

for which the solution in terms of χ and τ_{inv} are given in (23) and (24). Note that in this special case χ and τ_{inv} can also be expressed in terms of geometrical functions

$$\chi_{AE\Psi} = \frac{3}{2}\omega\frac{\rho^2}{\gamma_1^2}\left(\exp\left(\frac{2}{\rho}\frac{A'/A}{|\Theta'/\Theta|^2}\right)\sin\left(-\frac{2}{\rho}\frac{\phi'}{|\Theta'/\Theta|^2}\right)\right)^{-1}, \tag{34}$$

and

$$\tau_{AE\Psi} = \omega\left(\tan\left(-\frac{2}{\rho}\frac{\phi'}{|\Theta'/\Theta|^2}\right)\right)^{-1}. \tag{35}$$

In the appendix, also a similar relationship is derived based on the asymptotic expansions of Bessel functions. This approximation only differs from (34) by the factor γ_1. The comparison of the approximations with and without γ_1 shows that using the factor γ_1 improves the result, thus only (33) is presented here.

Due to the digamma function it is not possible when $V \neq 0$ to rewrite (29) such that the diffusivity χ can be estimated explicitly. Therefore, another approximation step is

introduced when $V \neq 0$ by replacing $F(a) = F\left(\frac{1}{2}\right)$. Then, it is still possible to estimate χ well in cases where $a \approx \frac{1}{2}$, e.g., $V \approx 0$ or $\ln\left((\lambda_2 - \lambda_1)\rho\right) \gg F(a)$. The reason is that V is also contained in $\lambda_2 - \lambda_1$. In that case (29) is expressed as

$$\left(\frac{V}{2\chi}\right)^2 + \frac{3}{2}\frac{\tau_{inv}}{\chi} + \frac{3}{2}\frac{i\omega}{\chi} = \frac{\gamma_1^2}{\rho^2}\exp\left(\left(\frac{\Theta'}{\Theta}\rho\right)^{-1}\right)^2. \tag{36}$$

Interestingly, the estimate (34) for the diffusivity χ does not change as the imaginary part only contains χ. If $\tau_{inv} = 0$ in (36), then

$$|V_{AE\Psi}| = \frac{3}{2}2\omega\frac{\sqrt{\Re\{z^2\}}}{\Im\{z^2\}}. \tag{37}$$

However, the possibility to estimate V in practice using (37) is questionable. The combination of V and τ_{inv} cannot be calculated using this approximation.

C. Multiple harmonics

Every harmonic fixes two degrees of freedom, which means in practice that either χ and τ_{inv} or χ and V can be estimated if only one harmonic is used. Therefore, to estimate χ, V, and τ_{inv} together, it is necessary to use at least two harmonics. In [1] an approximation is derived based on slab-geometry. Here, an approximation is derived based on the extended model in [4]

$$\chi = \frac{3}{4}\omega\left(\left(\frac{A'}{A} + \frac{1}{2\rho} + \frac{n'}{2n} + \frac{\chi'}{2\chi} + \frac{V}{2\chi}\right)\phi' + \frac{\phi''}{2}\right)^{-1} \tag{38}$$

This relation is derived by substituting $\Theta = A\exp(i\phi)$ and its spatial derivatives Θ' and Θ'' into the Laplace transform of (1), which also includes the density (assumed constant outside this subsection). Generally, this model is simplified by assuming constant χ such that $\chi'/2\chi = 0$. In addition, it is assumed that $V = 0$ and $\phi'' = 0$. However, it can be shown, by taking the spatial derivative of (10), that $\phi'' \neq 0$. On the other hand, without this assumption the relationship (38) cannot be used without approximating ϕ'' directly.

In this paper, only the assumption $\phi'' = 0$ is retained, but by using two harmonics the term $\frac{\chi'}{2\chi} + \frac{V}{2\chi}$ can be eliminated. As such, it is possible to calculate χ, V, and τ_{inv}. This should make it possible to identify χ under influence of V and τ_{inv}, accepting errors caused by the $\phi'' = 0$ assumption. Therefore, rewrite (38) in terms of the unknown quantities

$$\chi\left(\frac{A'}{A} + \frac{1}{2\rho} + \frac{n'}{2n}\right) - \frac{3}{4}\frac{\omega}{\phi'} = -\chi\left(\frac{\chi'}{2\chi} + \frac{V}{2\chi}\right). \tag{39}$$

This gives for two harmonics ω_1 and ω_2 at the same ρ

$$\chi\left(\frac{A'_{\omega_1}}{A_{\omega_1}} + \frac{1}{2\rho} + \frac{1}{2}\frac{n'}{n}\right) - \frac{3}{4}\frac{\omega_1}{\phi'_{\omega_1}} =$$

$$\chi\left(\frac{A'_{\omega_2}}{A_{\omega_2}} + \frac{1}{2\rho} + \frac{1}{2}\frac{n'}{n}\right) - \frac{3}{4}\frac{\omega_2}{\phi'_{\omega_2}}, \tag{40}$$

where the short-hand notation $\phi'_{\omega_2} = \phi'(\omega_2)$ and $A'_{\omega_2}/A_{\omega_2} = A'(\omega_2)/A(\omega_2)$ is used. Solving (40) yields

$$\chi_{c2H} = \frac{3}{4}\frac{\frac{\omega_1}{\phi'(\omega_1)} - \frac{\omega_2}{\phi'(\omega_2)}}{\left(\frac{A'}{A}(\omega_1) - \frac{A'}{A}(\omega_2)\right)}. \tag{41}$$

Note that (41) is insensitive to density gradients. The convectivity V is found by rewriting (39) and next to assuming $\phi'' = 0$ also assuming $\chi' = 0$

$$V_{c2H} = -2\left(\frac{A'}{A} + \frac{1}{2\rho} + \frac{1}{2}\frac{n'}{n}\right)\chi_{c2H} + \frac{3}{2}\frac{\omega}{\phi'}, \tag{42}$$

where ω can be freely chosen, but ω_1 is preferable. However, calculating the damping is more difficult. Therefore, after the substitution of Θ' and Θ'' into (1) the real part should be considered

$$\chi\left(\frac{A''}{A} - (\phi')^2\right) = \frac{3}{2}\tau_{inv} - \left(\frac{1}{\rho}V + V' + \frac{n'}{n}V\right)$$

$$- \left\{\frac{1}{\rho}\chi + \frac{n'}{n}\chi + V\right\}\frac{A'}{A}. \tag{43}$$

This can be rewritten in terms of τ_{inv}

$$\tau_{c2H} = \frac{2}{3}\chi\left(\left(\frac{A''}{A} - (\phi')^2\right)\right)$$

$$+ \left(\frac{1}{\rho}\frac{V}{\chi} + \frac{V'}{\chi} + \frac{n'}{n}\frac{V}{\chi}\right) + \left\{\frac{1}{\rho} + \frac{n'}{n} + \frac{V}{\chi}\right\}\frac{A'}{A}\right). \tag{44}$$

After the substitution of (39) it yields

$$\tau_{c2H} = \frac{2}{3}\chi\left(\left(\frac{A''}{A} - (\phi')^2\right) + \left(\frac{1}{\rho}\frac{V}{\chi} + \frac{V'}{\chi} + \frac{n'}{n}\frac{V}{\chi}\right)\right.$$

$$\left. + \left\{\frac{1}{\rho} + \frac{n'}{n} + \left(\frac{3}{4}\frac{\omega}{\phi'\chi} - 2\left(\frac{A'}{A} + \frac{1}{2\rho} + \frac{n'}{2n}\right)\right)\right\}\frac{A'}{A}\right). \tag{45}$$

In this equation, still two unknowns are present, thus in the spirit of neglecting ϕ'', $\frac{\partial}{\partial\rho}\left(\frac{A'}{A}\right)$ is also neglected. In addition, the standard assumption of V is constant in space is used such that $V' = 0$. This results in the following relationship

$$\tau_{c2H} = \frac{2}{3}\chi\left(\left(\frac{A'}{A}\right)^2 - (\phi')^2 + \left(\frac{1}{\rho}\frac{V}{\chi} + \frac{n'}{n}\frac{V}{\chi}\right)\right.$$

$$\left. + \left\{\frac{1}{\rho} + \frac{n'}{n} + \left(\frac{3}{4}\frac{\omega}{\phi'\chi} - 2\left(\frac{A'}{A} + \frac{1}{2\rho} + \frac{n'}{2n}\right)\right)\right\}\frac{A'}{A}\right). \tag{46}$$

The calculation of τ_{inv} is based on a complex relationship. In addition, assuming $\phi'' = 0$ and $\frac{\partial}{\partial\rho}\left(\frac{A'}{A}\right) = 0$ introduces additional errors when estimating χ, V, and τ_{inv}. This influences the approximations τ_{c2H} and V_{c2H} more significantly and hence are not so useful in practice.

It may seem that assuming χ constant is no longer necessary by eliminating χ' in (41). However, if χ' is non-zero it also modifies ϕ''. As such, it is questionable if better estimates of χ can be achieved. The density gradient is still included here to show that it is contained. However, in this paper the density is assumed constant such that $n'/n = 0$.

The methods of continued fractions, asymptotic expansions, and multiple harmonics are used to derive many different explicit approximations of χ. They are all approximations and all have a different region in which their approximation is good and regions in which it is poor. The other explicit approximations are presented in the appendix, which together with the approximations presented here are summarized and compared in the next section.

IV.   OUTWARD SOLUTIONS

In this section, the explicit approximations for χ in a semi-infinite cylindrical geometry with constant spatial parameter dependencies are summarized and compared. The compared approximations, based on (10) and (11), are presented in tabularized form in this section. In addition, the slab-geometry formula's in [1] are also considered here.

This section consists of four parts: the overview table with all the derived approximations; a presentation and discussion on the selection of the best approximations when only χ is considered; a similar discussion when χ and τ_{inv} are considered ($V = 0$); and when χ, V, and τ_{inv} are considered. The comparison is based on five parameters (ρ, ω, χ, V, and τ_{inv}) and is presented in terms of normalized transport coefficients, i.e., $\bar{\chi} = \chi/\omega$, $\bar{V} = V/\omega$, and

$\bar{\tau}_{inv} = \tau_{inv}/\omega$. In case two harmonics are necessary, $A'/A(\omega_1)$, $A'/A(\omega_2)$, $\phi'(\omega_1)$, and $\phi'(\omega_2)$ are calculated using $\omega_1 = \omega$ and $\omega_2 = 2\omega$ corresponding to the first and second harmonic.

A.   Overview of possible explicit approximations

In Table I the derived approximations to calculate χ from Section III and the appendix are summarized including some relationships from the literature. The following notation is used to express approximations based on continued fractions χ_{Kj3}: K denotes the Bessel function of the second kind, small j denotes the J-fraction, and 3 denotes the approximation order (truncation). The approximations derived from asymptotic expansions are denoted as follows: $\chi_{AE\Psi}$, where AE means Asymptotic Expansion and Ψ the Confluent Hypergeometric Function of the second kind on which it is based.

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	χ	Equation for χ	Equation for V	Equation for τ_{inv}	ref.
s	χ_{s1}	$\frac{3}{4}\frac{\omega}{(\phi')^2}$	0	0	[18]
	χ_{s2}	$\frac{3}{4}\frac{\omega}{(A'/A)^2}$	0	0	[19]
l	χ_{s3}	$\frac{3\omega}{(A'/A+\phi')^2}$	0	0	[20]
a	χ_{s4}	$\frac{3}{4}\frac{\omega}{(A'/A)\phi'}$	0	$\frac{\omega}{2}\left(\frac{A'/A}{\phi'}-\frac{\phi'}{A'/A}\right)$	[21]
b	χ_V	$\frac{3}{2}\frac{\omega\frac{A'}{A}}{\left(\left(\frac{A'}{A}\right)^2+(\phi')^2\right)\phi'}$	$\frac{3}{2}\frac{\omega\left((\phi')^2-\left(\frac{A'}{A}\right)^2\right)}{\left(\left(\frac{A'}{A}\right)^2+(\phi')^2\right)\phi'}$	0	[1]
	χ_ϕ	$\frac{3}{4}\sqrt{\frac{(\omega_1\phi'_{\omega_2})^2-(\omega_2\phi'_{\omega_1})^2}{\phi'^2_{\omega_1}\phi'^2_{\omega_2}(\phi'^2_{\omega_1}-\phi'^2_{\omega_2})}}$	$-2\chi\frac{A'}{A}-\sqrt{\frac{C_V+\sqrt{C_V^2+36\chi^2\omega^2}}{2}}$	$\frac{C_V-V^2}{6\chi}$	[1]
			$C_V=\frac{9}{4}\omega^2(\phi')^{-2}-4\chi^2(\phi')^2$		
c	χ_c	$\frac{3}{4}\frac{\omega}{\phi'\left(\frac{A'}{A}+\frac{1}{2\rho}\right)}$	0	$\frac{\omega}{2}\left(\frac{\left(\frac{A'}{A}+\frac{1}{2\rho}\right)}{\phi'}-\frac{\phi'}{\left(\frac{A'}{A}+\frac{1}{2\rho}\right)}\right)$	[4]
y	χ_{Kj1A}	$\frac{3}{4}\frac{\omega}{\left(\frac{A'}{A}+\frac{1}{2\rho}\right)^2}$	0	0	(54)
l	χ_z	$\frac{3}{2}\frac{\omega}{\Im(z^2)}$	0	$\omega\frac{\Re(z^2)}{\Im(z^2)}$	(23)
	χ_{c2H}	$\frac{3}{4}\frac{\omega_1/\phi'_{\omega_1}-\omega_2/\phi'_{\omega_2}}{\left(A'_{\omega_1}/A_{\omega_1}-A'_{\omega_2}/A_{\omega_2}\right)}$	$-2\chi\left(\frac{A'}{A}+\frac{1}{2\rho}\right)+\frac{3}{2}\frac{\omega}{\phi'}$	$\frac{2}{3}\chi\left(\left(\frac{A'}{A}\right)^2-(\phi')^2+\frac{1}{\rho}\frac{A'}{A}+\frac{V}{\chi}\frac{A'}{A}\right)$	(41)

Table I: Overview of approximations for χ based on a semi-infinite domain where heat waves are traveling towards the edge in a cylindrical geometry and slab geometry. From left to right, the columns denote: the geometry on which it is based; the approximation of χ either explicit or in terms of z in which case Table II gives the relationship for z; and the equations for V and τ_{inv}. In the last column, the reference of the origin of the equation are given. The short-hand notations $\phi'(\omega_1)=\phi'_{\omega_1}$ and $\frac{A'(\omega_1)}{A(\omega_1)}=\frac{A'_{\omega_1}}{A_{\omega_1}}$ are used. In slab-geometry the spatial derivatives are linear such that between two measurement points ρ_1 and ρ_2 the spatial derivatives are defined as $A'/A=\ln(A_2/A_1)/(\rho_2-\rho_1)$ and $\phi'=(\phi_2-\phi_1)/(\rho_2-\rho_1)$, where $\rho_2>\rho_1$, see [1] for details.

Many equations can also be expressed using polynomials in terms of z, Θ'/Θ, and ρ to directly calculate χ and τ_{inv}. Therefore, Table II gives the coefficients of these polynomials with their solutions to calculate z. Note that the approximations for τ_{inv} and V in Table I can show large deviations from the true values. Therefore, they should be used carefully, and in this paper they are only used in the validation process.

χ_z	Equation for z	Eq.
χ_{AEK}	$z = \rho^{-1} \exp\left(\left(\rho\frac{\Theta'}{\Theta}\right)^{-1}\right)$	(34)
$\chi_{AE\Psi}$	$z = \frac{\gamma_1}{\rho} \exp\left(\left(\frac{\Theta'}{\Theta}\rho\right)^{-1}\right)$	(33)
χ_{Kc2}	$b_2 = 4\rho, \quad b_1 = 3 + 4\frac{\Theta'}{\Theta}\rho, \quad b_0 = \frac{\Theta'}{\Theta}$	(48)
χ_{Kj2}	$b_2 = 8\rho^2, \quad b_1 = 8\rho^2\frac{\Theta'}{\Theta} + 4\rho, b_0 = 8\frac{\Theta'}{\Theta}\rho - 3$	(48)
	$z = \left(-b_1 + \sqrt{b_1^2 - 4\,b_0\,b_2}\right)/b_2$	
χ_{Kj3}	$a_3 = 16\rho^3, \quad a_2 = 16\frac{\Theta'}{\Theta}\rho^3 + 56\rho^2$ $a_1 = 48\frac{\Theta'}{\Theta}\rho^2 + 45\rho, \quad a_0 = 23\frac{\Theta'}{\Theta}\rho + 7.5$	(21)
χ_{Kc5}	$a_3 = 16\rho^2, \quad a_2 = \left(36\rho + 16\rho^2\frac{\Theta'}{\Theta}\right)$ $a_1 = \left(15 + 28\rho\frac{\Theta'}{\Theta}\right), \quad a_0 = 3\frac{\Theta'}{\Theta}$	(51)
	$p_2 = -27a_0a_3^2 + 9a_1a_2a_3 - 2a_2^3, p_0 = 3a_1a_3 - a_2^2$ $p_1 = \sqrt[3]{p_2 + \sqrt{4p_0^3 + p_2^2}}, z = \frac{1}{a_3}\left(-\frac{a_2}{3} - \frac{\sqrt[3]{2}p_0}{3p_1} + \frac{p_1}{3\sqrt[3]{2}}\right)$	

Table II: Overview of approximations for χ in terms of z for heat waves traveling towards the edge in a cylindrical geometry where an infinite domain is assumed. This table denotes the coefficients to calculate z using $\Theta'/\Theta = A'/A + i\phi'$ and ρ, which is used to calculate $\chi = \frac{3}{2}\omega/\Im\left(z^2\right)$ and $\tau_{inv} = \omega\Re\left(z^2\right)/\Im\left(z^2\right)$ based on the definition of z in (13). The equation numbers refer either to Section III or the appendix.

The different approximations for χ in this table are compared in the next subsection.

B.   Diffusivity only

The comparison for χ only ($V = 0$ and $\tau_{inv} = 0$) is made based on a large number of possibilities of χ, ω, and ρ.

There are different methods to find ϕ' and A'/A, which can influence the results significantly. Therefore, to avoid any ambiguity on how the derivatives ϕ' and A'/A should be approximated and how they affect the comparison, the true derivatives are used instead, which are generated by (11). The most interesting and best approximations are shown in Fig. 1 in terms of the relative error with respect to the true diffusivity χ.
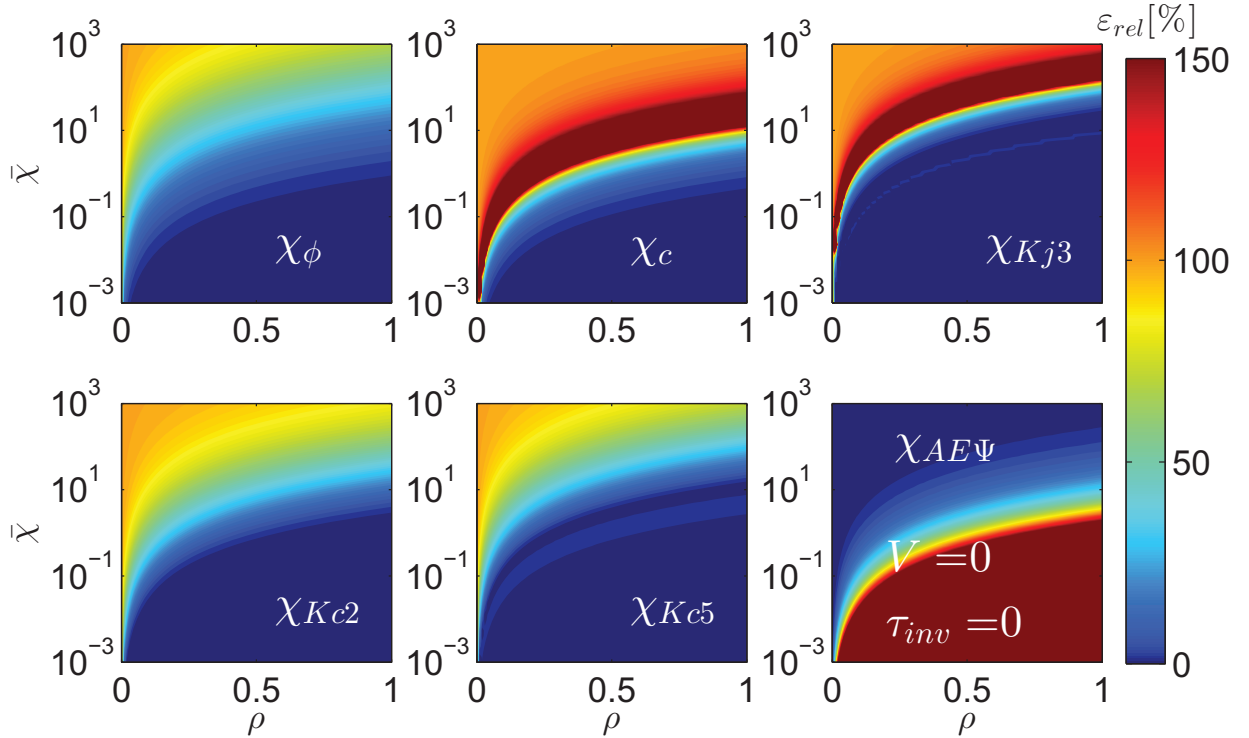
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Figure 1: Comparison between the different relative errors of the χ estimates for a large range of $\bar{\chi} = \chi/\omega$ and ρ. The relative error is defined as $\varepsilon_{rel} = 100 \times \frac{|\chi - \chi_{est}|}{\chi}$ [%] , where χ_{est} is either χ_ϕ (2 harmonics are used, which are $\omega_1 = \omega$ and $\omega_2 = 2\omega$), χ_c, χ_{Kj3}, χ_{Kc2}, χ_{Kc5}, and $\chi_{AE\Psi}$ from Table I. This comparison is based on a cylindrical geometry using an infinite domain boundary condition with χ and $V = \tau_{inv} = 0$, where the heat waves travel outwards. The darkest blue represents $\varepsilon_{rel} < 1\%$ and the darkest red represents all $\varepsilon_{rel} > 150\%$.

The approximations χ_ϕ, χ_c, χ_{Kj3}, χ_{Kc2}, and χ_{Kc5} are basically extensions of the slab-geometry case as they estimate χ well if the ratio $\omega\rho/\chi$ is large. Perturbations with large ω or small χ penetrate less deep. This also holds for errors and the influence of cylindrical geometry, which allows χ to be estimated well by χ_ϕ, χ_c , χ_{Kj3}, χ_{Kc2}, and χ_{Kc5}.

It is important to remember that χ_ϕ is based on two phases, making it less comparable to the other approximations. In χ_c and χ_{Kj3} large relative errors are observed for small ρ and large $\bar{\chi}$, which can be understood by considering χ_c in (17). This large error is caused by $(2\rho)^{-1}$ in χ_c, which over compensates resulting in a higher estimated diffusivity [4]. In addition, A'/A and ϕ' are negative quantities for heat waves traveling towards the edge, hence the sum of $(2\rho)^{-1}$ and A'/A results in zero at the center of the dark red area. On the other hand, the large errors are more difficult to show intuitively for the approximation

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χ_{Kj3}. The approximation χ_{Kj3} is more accurate and extends the region in which χ can be estimated well, which is a logical consequence of taking more terms in the continued fraction before truncation. Clearly, χ_{Kc2} and χ_{Kc5} do not suffer from the zero crossing causing the large errors in χ_c and χ_{Kj3}. The region in which χ is estimated well by χ_{Kc5} is slightly larger than χ_c and smaller than χ_{Kj3}.

Generally, χ_{Kj3} gives the best approximation of χ except when $\omega\rho/\chi$ is small. In that region the asymptotic expansions play an important role. They estimate χ well for $z\rho \approx 0$, which can be clearly seen in Fig. 1 for $\chi_{AE\Psi}$, which gave the best result of the two asymptotic expansions. This also means that if χ_{Kj3} and $\chi_{AE\Psi}$ are combined almost the entire region of χ is estimated well.

C.   Diffusivity and damping only

There are a number of suitable approximations to determine χ in the presence of τ_{inv}. However, it suffices with a minimal loss of accuracy to use the approximations χ_{Kj3} and $\chi_{AE\Psi}$ to estimate χ in the region of interest, which is shown in Fig. 2.
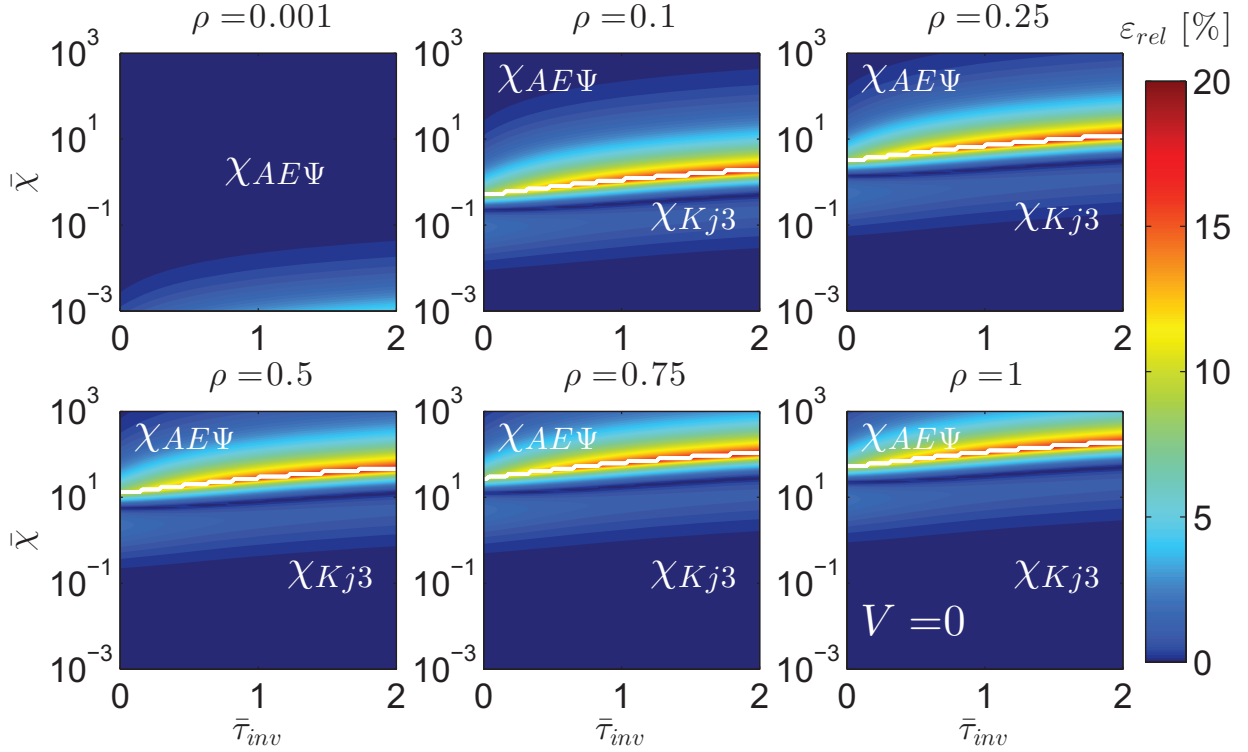
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Figure 2: Relative error of the χ estimates for the combination of χ_{Kj3} and $\chi_{AE\Psi}$ presented for different $\bar{\chi} = \chi/\omega$ and $\bar{\tau}_{inv} = \tau_{inv}/\omega$ represented at a number of spatial locations ρ. The relative error is defined as $\varepsilon_{rel} = 100 \times \frac{|\chi - \chi_{est}|}{\chi}$ [%]. This figure combines the approximations χ_{Kj3} and $\chi_{AE\Psi}$ which take both χ and τ_{inv} into account and are separated by the boundary represented by the white line (middle of the largest error). The resulting error is based on a cylindrical geometry using an infinite domain boundary condition with χ and $V = 0$, where the heat waves travel outwards.

The approximations are presented at a limited number of spatial locations ρ. In order to have significant impact on the heat pulse propagation, τ should be of the order of the energy confinement time (τ_e), i.e., 1 s for JET or ITER. Therefore, the range of τ is chosen such that $0.5 < \tau < \infty$ ($\tau = \infty$ meaning no damping), i.e., $0 \leqslant \tau_{inv} \leqslant 2$. This range is the same for the normalized $\bar{\tau}_{inv}$ as the applicable range of ω is assumed $\omega > 1$ [rad/s].

The white line shows the approximate boundary of the applicability of χ_{Kj3} and $\chi_{AE\Psi}$. This does not mean that χ_{Kj3} and $\chi_{AE\Psi}$ estimate χ with the highest accuracy compared to the other approximations in the presented region, but they have the largest region of approximation and are the most accurate in a large region. In addition, around the white line where the errors are largest χ_{Kj3} and $\chi_{AE\Psi}$ are the best approximations, which can also

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be understood by considering Fig. 1 again.

In general the effect of damping τ_{inv} acts as a shift parameter and is not directly influenced by the cylindrical geometry [1]. However, these effects are also influenced by the approximation. This means that for large τ_{inv} the regions in which χ are estimated well is extended for χ_{Kj3}, and similarly for χ_{ϕ}, χ_c, χ_{Kc2}, and χ_{Kc5}. On the other hand, it is reduced for $\chi_{AE\Psi}$, which is also shifted in the same direction.

The maximum relative error for the combination of $\chi_{AE\Psi}$ and χ_{Kj3} over the entire presented region is $\varepsilon_{rel} < 20\%$. It is also important to note that in some regions χ_{Kc2}, χ_{Kc5}, and χ_{Kj2} give a slightly better approximation. In some isolated regions the other approximations than the previously mentioned approximations, give a better approximation than the two presented in Fig. 2, but generally with a comparable accuracy. Note that the absolute error for large $\bar{\chi}$ will be larger than 20%.

D.   Diffusivity and convectivity with $\tau_{inv} = 0$ and $\tau_{inv} = 2$

Most of the previously discussed approximations poorly estimate the diffusivity in the presence of convectivity. In principle, only four relationships are available to analyze χ for heat waves traveling outwards, i.e., χ_V, χ_{ϕ}, $\chi_{AE\Psi}$, and χ_{c2H} (see Table I). It is unclear what a good range is for the parameter $\bar{V} = V/\omega$ except that it can also be negative. Therefore, an arbitrary choice for this range is made $-100 \leqslant \bar{V} \leqslant 100$.

It turns out that χ_{ϕ} outperforms all other approximations except $\chi_{AE\Psi}$ in a more stronger cylindrical geometry. On the other hand, χ_{c2H} performs similar to χ_{ϕ} both in error and the region in which it estimates χ well, making them almost interchangeable. Although they are both based on (the first) two harmonics, χ_{ϕ} only uses the phase. Therefore, only χ_{ϕ} and $\chi_{AE\Psi}$ are presented in Fig. 3 for $\tau_{inv} = 0$ and Fig. 4 for $\tau_{inv} = 2$.
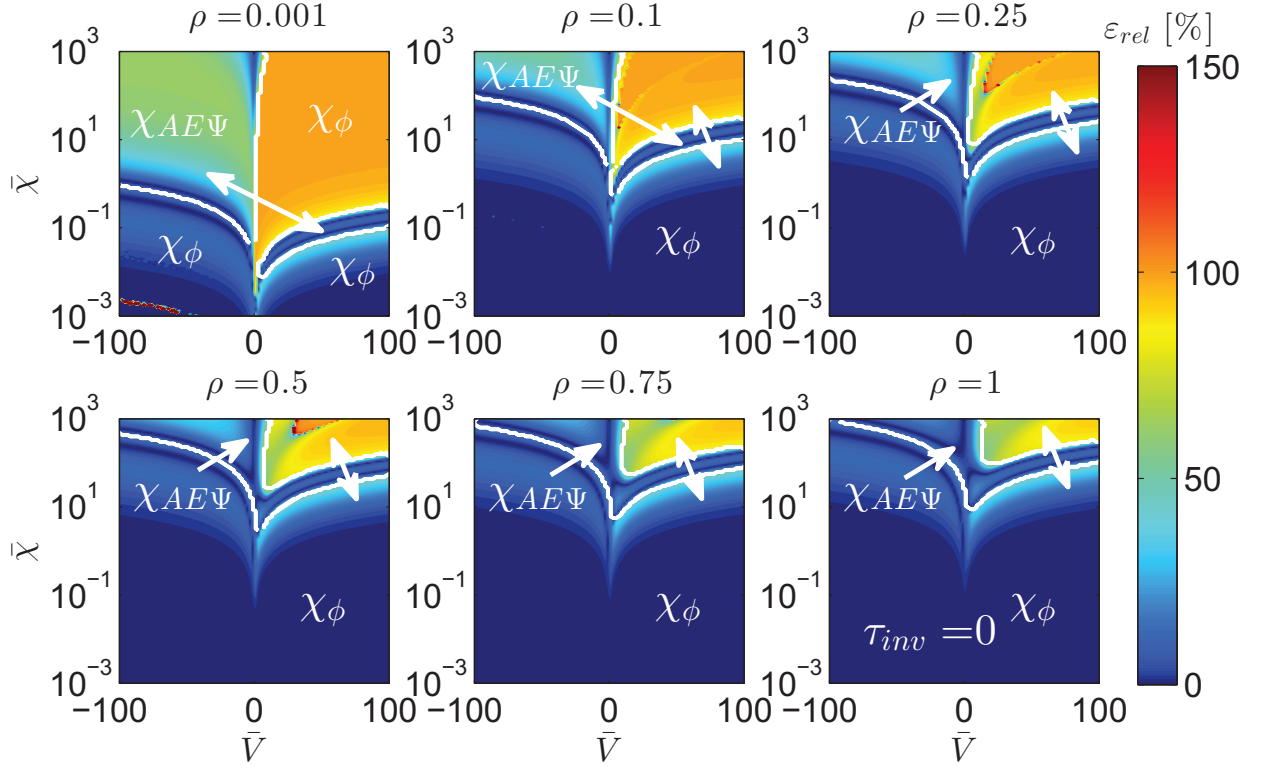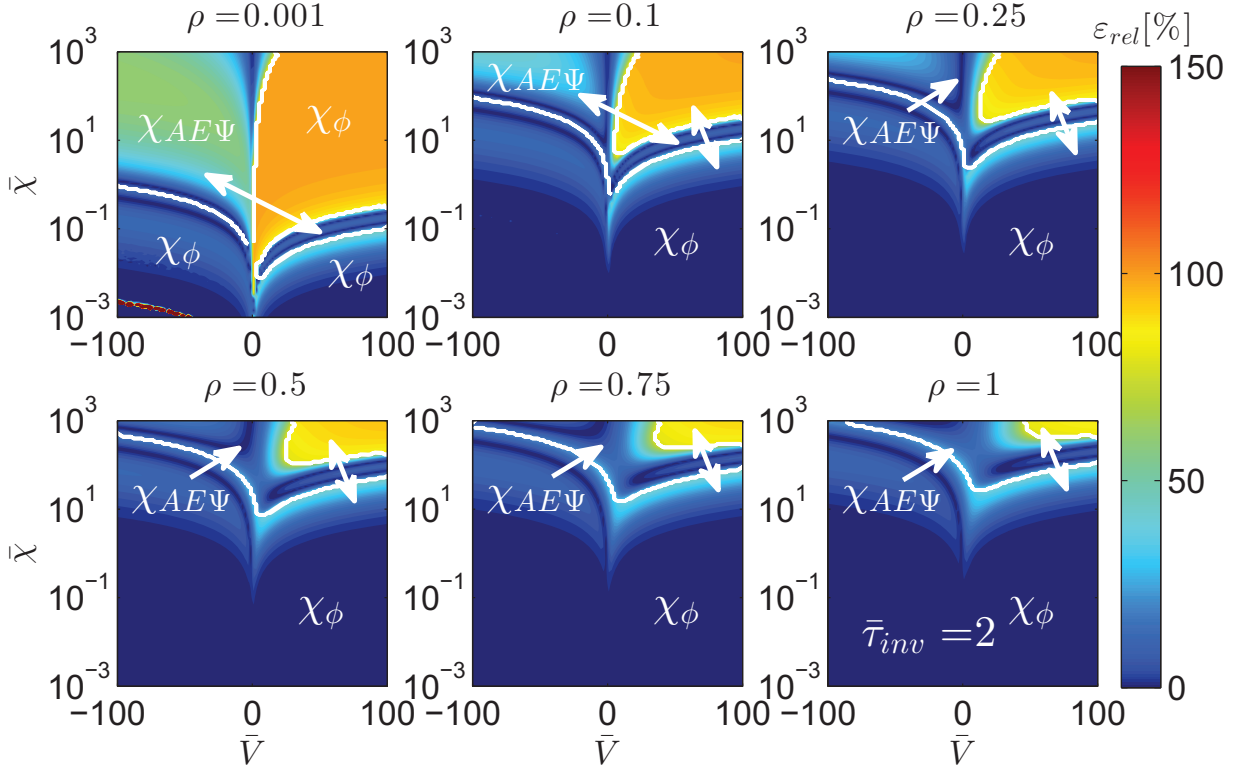
Figure 3: Relative error of the χ estimates for the combination of χ_ϕ and $\chi_{AE\Psi}$ presented for different $\bar{\chi} = \chi/\omega$ and $\bar{V} = V/\omega$ represented at a number of spatial locations ρ. The relative error is defined as $\varepsilon_{rel} = 100 \times \frac{|\chi - \chi_{est}|}{\chi}$ [%]. The approximation regions of χ_ϕ and $\chi_{AE\Psi}$ are separated by white lines. This comparison is based on a cylindrical geometry using an infinite domain boundary condition with $\tau_{inv} = 0$, where the heat waves travel outwards. The darkest blue represents $\varepsilon_{rel} < 1\%$ and the darkest red represents all $\varepsilon_{rel} > 150\%$.

Figure 4: Relative error of the χ estimates for the combination of χ_ϕ and $\chi_{AE\Psi}$ presented for different $\bar{\chi} = \chi/\omega$ and $\bar{V} = V/\omega$ represented at a number of spatial locations ρ. The relative error is defined as $\varepsilon_{rel} = 100 \times \frac{|\chi - \chi_{est}|}{\chi}$ [%]. The approximation regions of χ_ϕ and $\chi_{AE\Psi}$ are separated by white lines. This comparison is based on a cylindrical geometry using an infinite domain boundary condition with $\bar{\tau}_{inv} = 2$, where the heat waves travel outwards. The darkest blue represents $\varepsilon_{rel} < 1\%$ and the darkest red represents all $\varepsilon_{rel} > 150\%$.

Fig. 3 and Fig. 4 show similar regions. In general χ can be estimated well for large ρ, but there is an area with higher errors. Although this area is influenced by the damping, its effect is rather small. In addition, there is a small region which has large errors ($\varepsilon_{rel} > 150\%$) for large negative V and for small ρ and $\bar{\chi}$.

E.   Summary

In this section many different approximations are introduced. For problems with convective velocity, χ_ϕ and $\chi_{AE\Psi}$ should be used. Otherwise, $\chi_{AE\Psi}$, χ_ϕ, and χ_{Kj3} are the most important approximations. The other approximations can also be used. Although these

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other approximations are sometimes more accurate, their accuracy is in general comparable to the approximations given in Fig. 2 and Fig. 3. In Section V it is explained how to select the proper approximation based on the underlying models.

V.   CHOICE AND VALIDATION OF APPROXIMATIONS

In the previous section many different approximations have been presented. The validity ranges of these approximations do not only depend on ρ and ω, but also on the unknown transport coefficients χ, V, and τ_{inv}. This may seem to be a problem, but it is not. The reason is that the original models in terms of Bessel functions or Confluent Hypergeometric Functions are also available in (10) and (11). They can be used to explicitly calculate estimates of the original A'/A and ϕ' denoted by $\widehat{A'/A}$ and $\widehat{\phi}'$ given ρ and the transport coefficients, which should approximately be the same.

The procedure to select the approximation for the cases with only χ and τ_{inv} is shown through an example.

Example: Consider a measurement at $\rho = 0.1$ where $A'/A = -7.0593$ and $\phi' = -3.5588$ with $\omega = 50\pi$ and assuming that $V = 0$. Basically, two approximations are best tried first, i.e., χ_{Kj3} and $\chi_{AE\Psi}$. This results in $\chi_{Kj3} = 7.8629$, $\tau_{Kj3} = -5.0253$ and $\chi_{AE\Psi} = 13.1327$, $\tau_{AE\Psi} = 72.4163$ using Table II and Table I. Clearly, the τ_{inv}'s are erroneous. To test which is the proper approximation, substitute the values of χ and τ_{inv} back into the original model, which in this case is (11). This results in $\widehat{A'/A} + \widehat{\phi}'i = -6.8527 - 2.2746i$ for χ_{Kj3}, and $\widehat{A'/A} + \widehat{\phi}'i = -7.0377 - 3.6514i$ for $\chi_{AE\Psi}$. It is immediately clear that χ_{Kj3} is the best approximation as $\widehat{A'/A} \approx A'/A$ and $\widehat{\phi}' \approx \phi'$.

It is also possible to use χ_c and χ_{Kj2} resulting in $\chi_c = 10.7168$, $\tau_c = -90.2764$ and $\chi_{Kj2} = 7.9655$, $\tau_{Kj2} = -18.1707$, with $\widehat{A'/A} + \widehat{\phi}'i = -5.9559 - 4.1495i$ and $\widehat{A'/A} + \widehat{\phi}'i = -6.8924 - 3.7907i$, respectively. Again, χ_{Kj3} is closest to the true A'/A and ϕ' and hence should be selected.

In this example, the true values are $\chi = 8$ and $\tau_{inv} = 0.5$ such that actually χ_{Kj2} is closest to the true χ. However, due to the much larger error in τ_{Kj2} the $\widehat{A'/A}$ and $\widehat{\phi}'$ deviate significantly from the true A'/A and ϕ'.

Hence, it is the combination of χ and τ_{inv}, which is tested. Nevertheless, this test gives a clear statement of which χ estimate is trustworthy. In most cases the exact $A'/A = \widehat{A'/A} =$

and $\phi' = \widehat{\phi}'$ are found with some small rounding error, unlike the example presented here.

Similarly, the approximation can be tested for cases of approximations for χ only or for problems with χ and V only as only one harmonic is necessary. However, two harmonics are necessary for the mixed case of χ, V, and τ_{inv}. This procedure is more difficult. It is still possible to compare $\widehat{A'/A}$ and $\widehat{\phi}'$ to A'/A and ϕ', but this comparison also depends on A'/A and ϕ' of the second harmonic, which can also introduce a difference even if the estimated parameters are correct. Nevertheless, this comparison still gives valuable information on the quality of the approximation used. In case χ_ϕ or χ_V are used, the transformation in [1] can be used to calculate V and τ_{inv}.

If $V \neq 0$, then also an implementation of the Confluent Hypergeometric Function is required, which is not always available especially for complex valued arguments. In this paper, Mathematica$^{©}$ is used to calculate Ψ. Alternatively, it is also possible to use the continued fraction on p. 326 in [15], which can in principle be used to calculate Ψ with a desired accuracy using truncation. However, selecting a priori the number of terms necessary to arrive at a desired accuracy is not easy.

The procedure presented here checks the quality of the approximation with respect to its underlying model. It does not give any validation of the true χ as it depends on many other factors such as variation of the profiles, boundary conditions, approximation A'/A and ϕ', disturbances such as noise, and non-linearities. Noise and non-linearities require an extensive study based on statistics and higher harmonics, which is considered to be beyond the scope of this paper. The problem of varying profiles, approximation A'/A and ϕ', and boundary conditions is studied in [1].

VI. SUMMARY AND DISCUSSION

In this paper, the problem of determining the diffusion coefficient from measurements during power modulation experiments has been revisited. A large number of new approximations have been introduced to estimate χ directly from A'/A and ϕ' for different combinations of χ, V, and τ_{inv}. The approximations are based on infinite domains and are derived on the basis of cylindrical geometry using standard assumptions. These approximations, the approximations from the literature, and those of Part 1 of this paper series have been compared for heat waves traveling outwards (towards the edge).

The quality of the approximations is presented in several figures. In case only χ and τ_{inv} are considered ($V = 0$), the relative error of the χ estimate for the region of interest is in general 1%. However, in a small region the errors are larger with a maximum relative error of 20%. These errors are achievable by combining χ_{Kj3} and $\chi_{AE\Psi}$. In case also V is considered, the new approximations show a significant region in which χ can be estimated well, but also regions in which no suitable approximation exists. In this case, combining $\chi_{AE\Psi}$ and χ_ϕ covers a large region where χ can be well estimated.

Several figures present the ranges where χ can be estimated well as function of ρ, ω, and the transport coefficients. However, in practice these are less useful as the true transport coefficients are unknown. Therefore, in this paper a test is presented to check if the chosen approximation reproduces the original A'/A and ϕ'. This means that without using knowledge of the actual transport coefficients a proper approximation can be selected. Therefore, it is also necessary to determine the convectivity and damping.

In this paper, the estimation of χ in a cylindrical geometry using several approximations based on perturbative measurements for heat waves traveling outwards have been discussed. In Part 3, approximations to estimate χ are derived and discussed for heat waves traveling towards the center (inwards) using either semi-infinite domain assumptions or a symmetry boundary condition [22]. The latter is the natural boundary condition for a cylindrical geometry.

VII.   ACKNOWLEDGMENTS

1.   Approximations based on continued fractions

Continued J-fraction of the ratio of Bessel functions of the second kind

The continued J-fraction of the ratio between $K_1\left(z\right)/K_0\left(z\right)$ is given in (14) and is based on p. 364 in [15]. It is used to calculate three approximations of which two are given in (17) and (21). The approximation considering two terms is given by

$$\frac{\Theta'}{\Theta} = -z - \frac{1}{2\rho} + \frac{1}{\rho}\left(\frac{1/4}{2z\rho+2}\right), \tag{47}$$

where $z = \sqrt{\frac{3}{2}\frac{i\omega+\tau_{inv}}{\chi}}$ and Θ'/Θ is defined according to (9). It can be written in polynomial form, i.e.,

$$0 = 8\rho^2 z^2 + \left(8\rho^2\frac{\Theta'}{\Theta} + 4\rho\right)z + 8\frac{\Theta'}{\Theta}\rho - 3. \tag{48}$$

This polynomial is solved in terms of z to calculate χ (see Table I).

Continued C-fraction of the ratio of Bessel functions of the second kind

The following continued C-fraction of the ratio $K_1\left(z\right)/K_0\left(z\right)$ is based on p. 363 in [15]

$$\frac{K_1\left(z\rho\right)}{K_0\left(z\rho\right)} = \cfrac{1}{1 - \cfrac{\frac{1}{2z\rho}}{1 + \cfrac{\frac{3}{2}\frac{1}{2z\rho}\Big|_{\mathrm{I}}}{1 + \cfrac{\frac{1}{2}\frac{1}{2z\rho}}{1 + \cfrac{\frac{5}{2}\frac{1}{2z\rho}\Big|_{\mathrm{II}}}{1 + \cdots}}}}}. \tag{49}$$

It is used to approximate (11). Here, it is chosen to truncate only at locations I and II to reduce the number of approximations. If (49) is truncated at location I, this results in the following polynomial in terms of z

$$0 = 4\rho z^2 + \left(3 + 4\frac{\Theta'}{\Theta}\rho\right)z + \frac{\Theta'}{\Theta}, \tag{50}$$

which needs to be solved to find χ and τ_{inv}. In case (49) is truncated at location II, it gives the following polynomial, which can also be used to find χ and τ_{inv}

$$0 = 16\rho^2 z^3 + \left(36\rho + 16\rho^2\frac{\Theta'}{\Theta}\right)z^2 + \left(15 + 28\rho\frac{\Theta'}{\Theta}\right)z + 3\frac{\Theta'}{\Theta}. \tag{51}$$

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2.   Approximations for χ only ($V = \tau_{inv} = 0$)

Consider again (16)

$$z = -\left(\frac{A'}{A} + i\phi' + \frac{1}{2\rho}\right). \tag{16}$$

Substituting z and expanding

$$\frac{3}{2}\frac{i\omega}{\chi} = \left(\frac{A'}{A} + \frac{1}{2\rho}\right)^2 + 2i\phi'\left(\frac{A'}{A} + \frac{1}{2\rho}\right) - (\phi')^2. \tag{52}$$

If the imaginary part is rewritten in terms of χ, it results in

$$\chi_{s1} = \frac{3}{4}\frac{\omega}{(\phi')^2}. \tag{53}$$

This variation was also derived in [18] based on Bessel functions and can be derived in slab-geometry. The real part gives

$$\chi_{Kj1A} = \frac{3\omega}{4\left(\frac{A'}{A} + \frac{1}{2\rho}\right)^2}. \tag{54}$$

The continued fractions with more terms lead to a mix between amplitude and phase, which are complicated to solve. Hence, they are not calculated here.

3.   Asymptotic expansions based on the Bessel function of the second kind

From [17] the asymptotic expansion for $z\,\rho \approx 0$ and fixed integer order $\nu \neq 0$ is given by

$$K_\nu \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\,\rho\right)^{-\nu}, \tag{55}$$

and for order $\nu = 0$ is given by

$$K_0 \sim -\ln(z\,\rho). \tag{56}$$

These can be substituted into (11) to describe Θ'/Θ around $z\,\rho \approx 0$

$$\frac{\Theta'}{\Theta} = -z\frac{\left(\frac{1}{2}\Gamma(1)\left(\frac{1}{2}z\,\rho\right)^{-1}\right)}{(-\ln(z\,\rho))}. \tag{57}$$

Simplifying yields

$$\frac{\Theta'}{\Theta} = \frac{1}{\rho\ln(z\,\rho)}. \tag{58}$$

This can again be expressed in terms of z

$$z = \rho^{-1} \exp\left(\left(\rho \frac{\Theta'}{\Theta}\right)^{-1}\right), \tag{59}$$

which can be solved using the techniques given in Table I.

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