Chapter 5: Waves and characteristics

Overview

- **Physics and accounting:** use example of sound waves to illustrate method of linearization and counting of variables and solutions; [book: Sec. 5.1]
- **MHD waves:** different representations and reductions of the linearized MHD equations, obtaining the three main waves, dispersion diagrams; [book: Sec. 5.2]
- **Phase and group diagrams:** propagation of plane waves and wave packets, asymptotic properties of the three MHD waves; [book: Sec. 5.3]
- **Characteristics:** numerical method, classification of PDEs, application to MHD. [book: Sec. 5.4]

Sound waves

- Perturb the gas dynamic equations ($\mathbf{B} = 0$),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p = 0\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \tag{3}$$

about infinite, homogeneous gas at rest,

$$\begin{aligned} \rho(\mathbf{r}, t) &= \rho_0 + \rho_1(\mathbf{r}, t) \qquad (\text{where } |\rho_1| \ll \rho_0 = \text{const})\,, \\ p(\mathbf{r}, t) &= p_0 + p_1(\mathbf{r}, t) \qquad (\text{where } |p_1| \ll p_0 = \text{const})\,, \\ \mathbf{v}(\mathbf{r}, t) &= \qquad \mathbf{v}_1(\mathbf{r}, t) \qquad (\text{since } \mathbf{v}_0 = 0)\,. \end{aligned} \tag{4}$$

$\Rightarrow$ *Linearised equations of gas dynamics:*

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{5}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = 0\,, \tag{6}$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0\,. \tag{7}$$

Wave equation

- Equation for ρ_1 does not couple to the other equations: drop. Remaining equations give *wave equation for sound waves*:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - c^2\, \nabla\nabla \cdot \mathbf{v}_1 = 0\,, \tag{8}$$

where

$$c \equiv \sqrt{\gamma p_0/\rho_0} \tag{9}$$

is *the velocity of sound* of the background medium.

- Plane wave solutions

$$\mathbf{v}_1(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\mathbf{v}}_{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{10}$$

turn the wave equation (8) into an algebraic equation:

$$(\,\omega^2 \mathsf{I} - c^2\, \mathbf{k}\mathbf{k}\,) \cdot \hat{\mathbf{v}} = 0\,. \tag{11}$$

- For $\mathbf{k} = k\,\mathbf{e}_z$, the solution is:

$$\omega = \pm k\,c\,, \qquad \hat{v}_x = \hat{v}_y = 0\,, \quad \hat{v}_z \text{ arbitrary}\,, \tag{12}$$

$\Rightarrow$ *Sound waves* propagating to the right ($+$) and to the left ($-$): *compressible* ($\nabla\cdot\mathbf{v} \neq 0$) and *longitudinal* ($\mathbf{v} \parallel \mathbf{k}$) waves.

Counting

- There are also other solutions:

$$\omega^2 = 0\,, \qquad \hat{v}_x\,, \hat{v}_y \text{ arbitrary}\,, \qquad \hat{v}_z = 0\,, \tag{13}$$

  ⇒ *incompressible transverse ($\mathbf{v}_1 \perp \mathbf{k}$) translations.* They do not represent interesting physics, but simply establish completeness of the velocity representation.

- Problem: 1st order system (5)–(7) for ρ_1, $\mathbf{v}_1$, p_1 has 5 degrees of freedom, whereas 2nd order system (8) for $\mathbf{v}_1$ appears to have 6 degrees of freedom ($\partial^2/\partial t^2 \rightarrow -\omega^2$). However, the 2nd order system actually only has 4 degrees of freedom, since ω^2 does not double the number of translations (13). Spurious doubling of the eigenvalue $\omega = 0$ happened when we applied the operator $\partial/\partial t$ to Eq. (6) to eliminate p_1.

- Hence, we *lost one degree of freedom* in the reduction to the wave equation in terms of $\mathbf{v}_1$ alone. This happened when we dropped Eq. (5) for ρ_1. Inserting $\mathbf{v}_1 = 0$ in the original system gives the signature of this lost mode:

$$\omega\hat{\rho} = 0 \quad \Rightarrow \quad \omega = 0\,, \quad \hat{\rho} \text{ arbitrary}\,, \quad \text{but } \hat{\mathbf{v}} = 0 \ \text{ and } \ \hat{p} = 0\,. \tag{14}$$

  ⇒ *entropy wave*: perturbation of the density and, hence, of the entropy $S \equiv p\rho^{-\gamma}$. Like the translations (13), this mode does not represent important physics but is needed to account for the degrees of freedom of the different representations.

MHD waves

- Similar analysis for MHD in terms of ρ , $\mathbf{v}$, e $\left(\equiv \frac{1}{\gamma-1} p/\rho \right)$, and $\mathbf{B}$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{15}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + (\gamma - 1)\nabla(\rho e) + (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} = 0\,, \tag{16}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1) e \nabla \cdot \mathbf{v} = 0\,, \tag{17}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{v} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{18}$$

- *Linearise about plasma at rest,* $\mathbf{v}_0 = 0\,, \rho_0\,, e_0\,, \mathbf{B}_0 = \text{const}$:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{19}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)(e_0 \nabla \rho_1 + \rho_0 \nabla e_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 = 0\,, \tag{20}$$

$$\frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 = 0\,, \tag{21}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 = 0\,, \qquad \nabla \cdot \mathbf{B}_1 = 0\,. \tag{22}$$

Transformation

- *Sound* and vectorial *Alfvén speed*,

$$c \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}, \qquad \mathbf{b} \equiv \frac{\mathbf{B}_0}{\sqrt{\rho_0}}, \tag{23}$$

and dimensionless variables,

$$\tilde{\rho} \equiv \frac{\rho_1}{\gamma\, \rho_0}, \qquad \tilde{\mathbf{v}} \equiv \frac{\mathbf{v}_1}{c}, \qquad \tilde{e} \equiv \frac{e_1}{\gamma\, e_0}, \qquad \tilde{\mathbf{B}} \equiv \frac{\mathbf{B}_1}{c\sqrt{\rho_0}}, \tag{24}$$

⇒ *linearised MHD equations* with coefficients c and $\mathbf{b}$:

$$\gamma \frac{\partial \tilde{\rho}}{\partial t} + c\, \nabla \cdot \tilde{\mathbf{v}} = 0\,, \tag{25}$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + c\, \nabla \tilde{\rho} + c\, \nabla \tilde{e} + (\nabla \tilde{\mathbf{B}}) \cdot \mathbf{b} - \mathbf{b} \cdot \nabla \tilde{\mathbf{B}} = 0\,, \tag{26}$$

$$\frac{\gamma}{\gamma - 1} \frac{\partial \tilde{e}}{\partial t} + c\, \nabla \cdot \tilde{\mathbf{v}} = 0\,, \tag{27}$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} + \mathbf{b}\, \nabla \cdot \tilde{\mathbf{v}} - \mathbf{b} \cdot \nabla \tilde{\mathbf{v}} = 0\,, \qquad \nabla \cdot \tilde{\mathbf{B}} = 0\,. \tag{28}$$

Symmetry

- *Plane wave solutions*, with $\mathbf{b}$ and $\mathbf{k}$ arbitrary now:

$$\tilde{\rho} = \tilde{\rho}(\mathbf{r}, t) = \hat{\rho}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \quad \text{etc.} \tag{29}$$

yields an algebraic system of eigenvalue equations:

$$\begin{aligned} c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \gamma\,\omega\,\hat{\rho}\,, \\ \mathbf{k}\,c\,\hat{\rho} \qquad + \mathbf{k}\,c\,\hat{e} + (\mathbf{k}\mathbf{b}\cdot - \mathbf{k}\cdot\mathbf{b})\,\hat{\mathbf{B}} &= \omega\,\hat{\mathbf{v}}\,, \\ c\,\mathbf{k}\cdot\hat{\mathbf{v}} &= \tfrac{\gamma}{\gamma-1}\,\omega\,\hat{e}\,, \\ (\mathbf{b}\mathbf{k}\cdot - \mathbf{b}\cdot\mathbf{k})\,\hat{\mathbf{v}} &= \omega\,\hat{\mathbf{B}}\,, \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0\,. \end{aligned} \tag{30}$$

- $\Rightarrow$ **Symmetric eigenvalue problem!** (The equations for $\hat{\rho}$, $\hat{\mathbf{v}}$, $\hat{e}$, and $\hat{\mathbf{B}}$ appear to know about each other.) .

- The symmetry of the linearized system is closely related to an analogous property of the original nonlinear equations: *the nonlinear ideal MHD equations are symmetric hyperbolic partial differential equations*.

Matrix eigenvalue problem

- Choose $\mathbf{b} = (0, 0, b)\,, \quad \mathbf{k} = (k_\perp, 0, k_\parallel)$:

$$\begin{pmatrix} 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ k_\perp c & 0 & 0 & 0 & k_\perp c & -k_\parallel b & 0 & k_\perp b \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_\parallel b & 0 \\ k_\parallel c & 0 & 0 & 0 & k_\parallel c & 0 & 0 & 0 \\ 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} = \omega \begin{pmatrix} \gamma\,\hat\rho \\ \hat v_x \\ \hat v_y \\ \hat v_z \\ \frac{\gamma}{\gamma-1}\,\hat e \\ \hat B_x \\ \hat B_y \\ \hat B_z \end{pmatrix} . \tag{31}$$

$\Rightarrow$ **Another representation of the symmetry of linearized MHD equations.**

- New features of MHD waves compared to sound: occurrence of Alfvén speed b and anisotropy expressed by the two components $k_\parallel$ and $k_\perp$ of the wave vector. We could compute the dispersion equation from the determinant and study the associated waves, but we prefer again to exploit the much simpler velocity representation.

MHD wave equation

- Ignoring the magnetic field constraint $\mathbf{k} \cdot \hat{\mathbf{B}} = 0$ in the 8×8 eigenvalue problem (31) would yield *one spurious eigenvalue* $\omega = 0$. This may be seen by operating with the projector $\mathbf{k}\cdot$ onto Eq. (30)(d), which gives $\omega\, \mathbf{k} \cdot \hat{\mathbf{B}} = 0$.
- Like in the gas dynamics problem, a *genuine but unimportant marginal entropy mode* is obtained for $\omega = 0$ with $\hat{\mathbf{v}} = 0$, $\hat{p} = 0$, and $\hat{\mathbf{B}} = 0$:

$$\omega = 0\,, \qquad \hat{p} = \hat{e} + \hat{\rho} = 0\,, \qquad \hat{S} = \gamma \hat{e} = -\gamma \hat{\rho} \neq 0\,. \tag{32}$$

- Both of these marginal modes are eliminated by exploiting *the velocity representation.* The perturbations ρ_1, e_1, $\mathbf{B}_1$ are expressed in terms of $\mathbf{v}_1$ by means of Eqs. (19), (21), and (22), and substituted into the momentum equation (20). This yields the **MHD wave equation for a homogeneous medium**:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \left[(\mathbf{b} \cdot \nabla)^2\, \mathbf{I} + (b^2 + c^2)\, \nabla\nabla - \mathbf{b} \cdot \nabla\, (\nabla\, \mathbf{b} + \mathbf{b}\, \nabla) \right] \cdot \mathbf{v}_1 = 0\,. \tag{33}$$

The sound wave equation (8) is obtained for the special case $\mathbf{b} = 0$.

MHD wave equation (cont'd)

- Inserting plane wave solutions gives the required eigenvalue equation:

$$\Big\{ \big[\omega^2 - (\mathbf{k}\cdot\mathbf{b})^2 \big] \mathsf{I} - (b^2 + c^2)\,\mathbf{k}\,\mathbf{k} + \mathbf{k}\cdot\mathbf{b}\,(\mathbf{k}\,\mathbf{b} + \mathbf{b}\,\mathbf{k}) \Big\} \cdot \hat{\mathbf{v}} = 0\,, \qquad (34)$$

or, in components:

$$\begin{pmatrix} -k_\perp^2(b^2+c^2) - k_\parallel^2 b^2 & 0 & -k_\perp k_\parallel c^2 \\ 0 & -k_\parallel^2 b^2 & 0 \\ -k_\perp k_\parallel c^2 & 0 & -k_\parallel^2 c^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} . \qquad (35)$$

Hence, a 3×3 symmetric matrix equation is obtained in terms of the variable $\hat{\mathbf{v}}$, with *quadratic eigenvalue* ω^2, corresponding to the original 6×6 representation with eigenvalue ω (resulting from elimination of the two marginal modes).

- Determinant yields the **dispersion equation:**

$$\det = \omega\,(\omega^2 - k_\parallel^2 b^2)\,\Big[\omega^4 - k^2(b^2+c^2)\,\omega^2 + k_\parallel^2 k^2 b^2 c^2 \Big] = 0 \qquad (36)$$

(where we have artificially included a factor ω for the marginal entropy wave).

Roots

1) **Entropy waves:**

$$\omega = \omega_E \equiv 0\,, \tag{37}$$

$$\hat{\mathbf{v}} = \hat{\mathbf{B}} = 0\,, \quad \hat{p} = 0\,, \quad \text{but} \quad \hat{s} \neq 0\,. \tag{38}$$

⇒ just perturbation of thermodynamic variables.

2) **Alfvén waves:**

$$\omega^2 = \omega_A^2 \equiv k_\parallel^2 b^2 \quad \rightarrow \quad \omega = \pm\omega_A\,, \tag{39}$$

$$\hat{v}_x = \hat{v}_z = \hat{B}_x = \hat{B}_z = \hat{s} = \hat{p} = 0\,, \quad \hat{B}_y = -\hat{v}_y \neq 0\,. \tag{40}$$

⇒ transverse $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ so that field lines follow the flow.

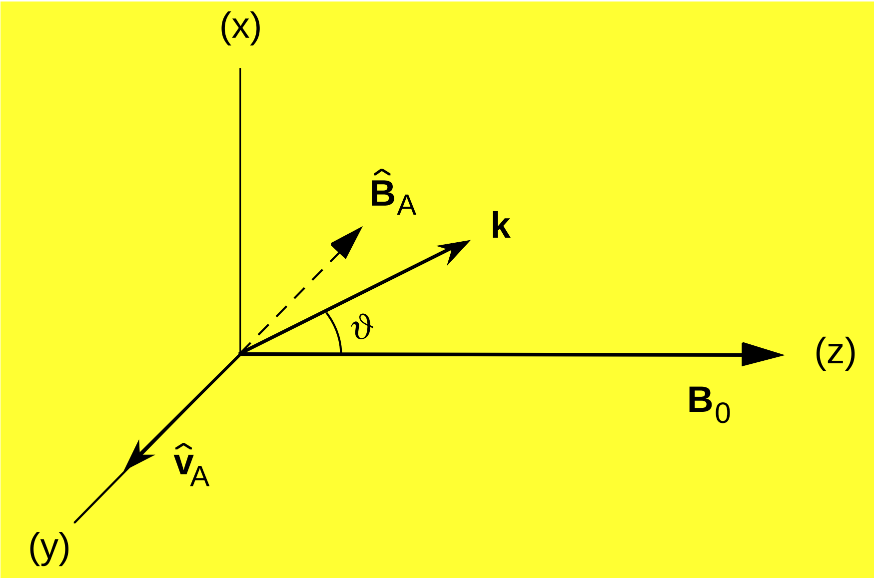
3) **Fast (+) and Slow (−) magnetoacoustic waves:**

$$\omega^2 = \omega_{s,f}^2 \equiv \tfrac{1}{2}k^2(b^2+c^2)\,\Big[\,1 \pm \sqrt{1 - \frac{4k_\parallel^2 b^2 c^2}{k^2(b^2+c^2)^2}}\,\Big] \quad \rightarrow \quad \omega = \begin{cases} \pm\omega_s \\ \pm\omega_f \end{cases} \tag{41}$$

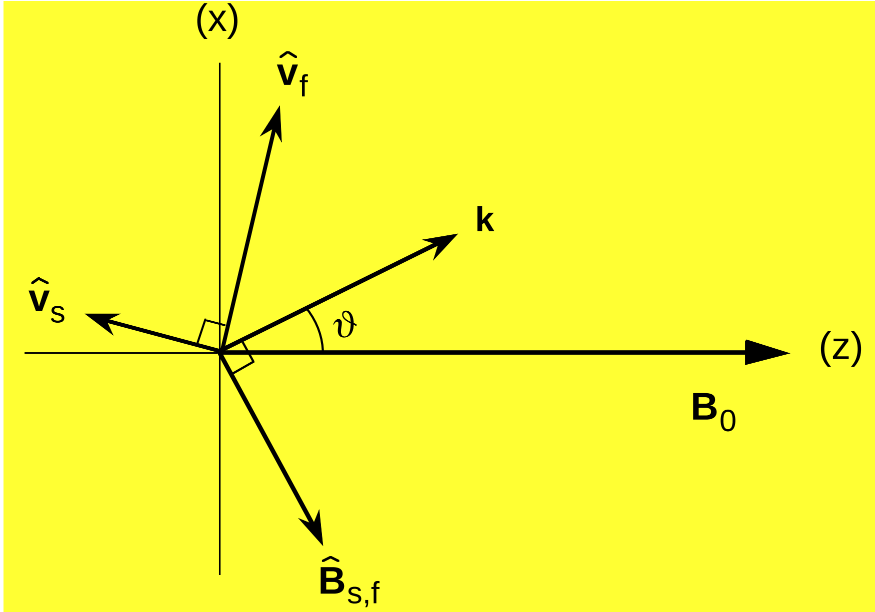
$$\hat{v}_y = \hat{B}_y = \hat{s} = 0\,, \quad \text{but} \quad \hat{v}_x\,, \hat{v}_z\,, \hat{p}\,, \hat{B}_x\,, \hat{B}_z \neq 0\,, \tag{42}$$

⇒ perturbations $\hat{\mathbf{v}}$ and $\hat{\mathbf{B}}$ in the plane through $\mathbf{k}$ and $\mathbf{B}_0$.

Eigenfunctions



Alfvén waves



Magnetosonic waves

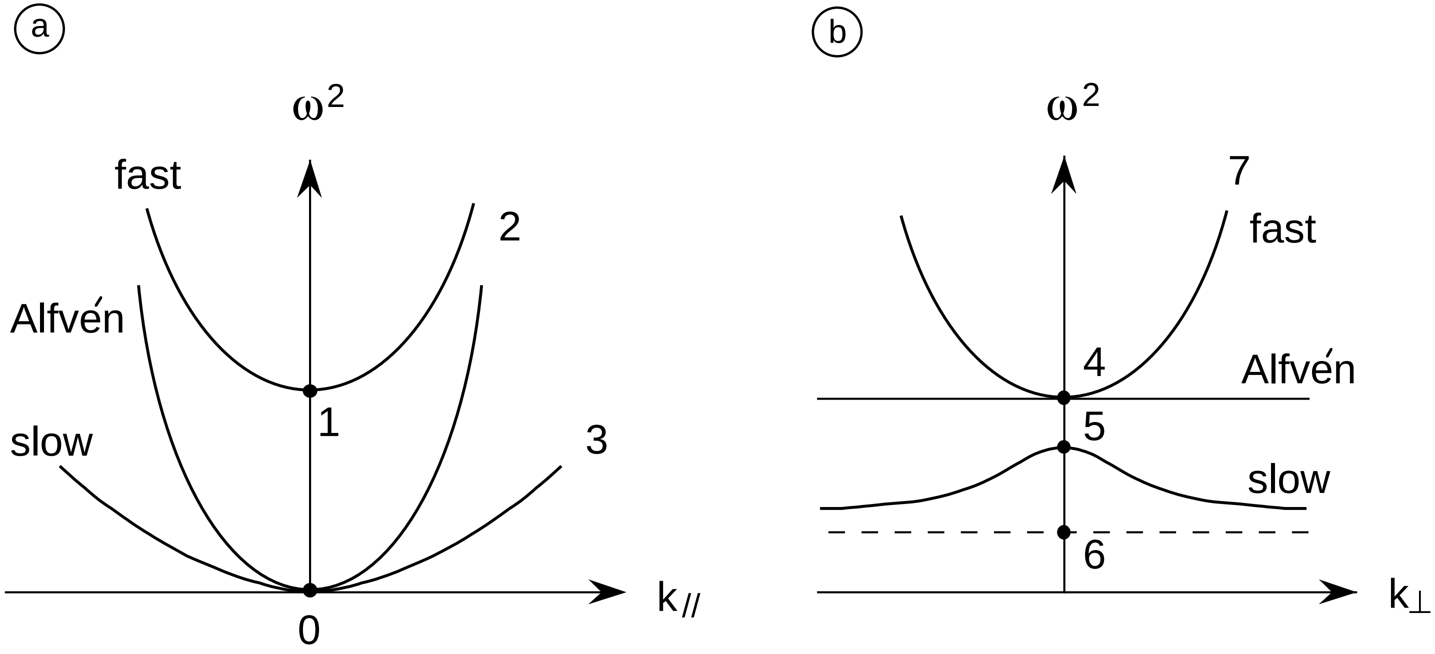
- Note: **the eigenfunctions are mutually orthogonal:**

$$\hat{\mathbf{v}}_s \perp \hat{\mathbf{v}}_A \perp \hat{\mathbf{v}}_f \,. \tag{43}$$

$\Rightarrow$ Arbitrary velocity field may be decomposed at all times (e.g. at $t = 0$) in the three MHD waves: **the initial value problem is a well-posed problem**.

Dispersion diagrams (schematic)

[exact diagrams in book: Fig. 5.3, scaling $\bar{\omega} \equiv (l/b)\,\omega\,,\ \bar{k} \equiv k\,l$]



- Note: $\omega^2(k_\parallel = 0) = 0$ for Alfvén and slow waves $\Rightarrow$ potential onset of *instability.*
- Asymptotics of $\omega^2(k_\perp \to \infty)$ characterizes *local* behavior of the three waves:

$$\left\{ \begin{array}{lll} \partial\omega/\partial k_\perp > 0\,, & \omega_f^2 \to \infty & \text{for fast waves,} \\ \partial\omega/\partial k_\perp = 0\,, & \omega_A^2 \to k_\parallel^2 b^2 & \text{for Alfvén waves,} \\ \partial\omega/\partial k_\perp < 0\,, & \omega_s^2 \to k_\parallel^2 \dfrac{b^2 c^2}{b^2 + c^2} & \text{for slow waves.} \end{array} \right. \qquad (44)$$

Phase and group velocity

Dispersion equation $\omega = \omega(\mathbf{k}) \;\Rightarrow\;$ two fundamental concepts:

1. A *single plane wave* propagates in the direction of $\mathbf{k}$ with the **phase velocity**

$$\mathbf{v}_{\rm ph} \equiv \frac{\omega}{k}\,\mathbf{n}\,, \qquad \mathbf{n} \equiv \mathbf{k}/k = (\sin\vartheta, 0, \cos\vartheta)\,; \qquad (45)$$

$\Rightarrow$ MHD waves are non-dispersive (only depend on angle ϑ, not on $|\mathbf{k}|$):

$$(\mathbf{v}_{\rm ph})_A \equiv b\cos\vartheta\,\mathbf{n}\,, \qquad (46)$$

$$(\mathbf{v}_{\rm ph})_{s,f} \equiv \sqrt{\tfrac{1}{2}(b^2+c^2)}\,\sqrt{1 \pm \sqrt{1-\sigma\cos^2\vartheta}}\;\mathbf{n}\,, \qquad \sigma \equiv \frac{4b^2c^2}{(b^2+c^2)^2}\,. \qquad (47)$$

2. A *wave packet* propagates with the **group velocity**

$$\mathbf{v}_{\rm gr} \equiv \frac{\partial\omega}{\partial\mathbf{k}} \quad \Big[\equiv \frac{\partial\omega}{\partial k_x}\mathbf{e}_x + \frac{\partial\omega}{\partial k_y}\mathbf{e}_y + \frac{\partial\omega}{\partial k_z}\mathbf{e}_z \Big]\,; \qquad (48)$$

$\Rightarrow$ MHD caustics in directions $\mathbf{b}$, and mix of $\mathbf{n}$ and $\mathbf{t}$ ($\perp\,\mathbf{n}$):

$$(\mathbf{v}_{\rm gr})_A = \mathbf{b}\,, \qquad (49)$$

$$(\mathbf{v}_{\rm gr})_{s,f} = (v_{\rm ph})_{s,f}\left[\mathbf{n} \pm \frac{\sigma\sin\vartheta\cos\vartheta}{2\sqrt{1-\sigma\cos^2\vartheta}\,\big[1 \pm \sqrt{1-\sigma\cos^2\vartheta}\,\big]}\,\mathbf{t}\right]. \qquad (50)$$

Wave packet

Wave packet of plane waves satisfying dispersion equation $\omega = \omega(\mathbf{k})$:

$$\Psi_i(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, d^3k \,. \tag{51}$$

Evolves from initial shape given by *Fourier synthesis,*

$$\Psi_i(\mathbf{r}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, d^3k \,, \tag{52}$$

where amplitudes $A_i(\mathbf{k})$ are related to initial values $\Psi_i(\mathbf{r}, 0)$ by *Fourier analysis,*

$$A_i(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \Psi_i(\mathbf{r}, 0)\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, d^3r \,. \tag{53}$$

MHD: Ψ_i – perturbations ($\tilde{\rho}_1$,) $\tilde{\mathbf{v}}_1$ (, $\tilde{e}_1$, $\tilde{\mathbf{B}}_1$); A_i – Fourier amplitudes ($\hat{\rho}_1$,) $\hat{\mathbf{v}}_1$ (, $\hat{e}_1$, $\hat{\mathbf{B}}_1$).

Example: Gaussian wave packet of harmonics centered at some wave vector $\mathbf{k}_0$,

$$A_i(\mathbf{k}) = \hat{A}_i\, e^{-\frac{1}{2}|(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{a}|^2} \,, \tag{54}$$

corresponds to initial packet with main harmonic $\mathbf{k}_0$ and modulated amplitude centered at $\mathbf{r} = 0$:

$$\Psi_i(\mathbf{r}, 0) = \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}} \times \frac{\hat{A}_i}{a_x a_y a_z}\, e^{-\frac{1}{2}[(x/a_x)^2 + (y/a_y)^2 + (z/a_z)^2]} \,. \tag{55}$$

Wave packet (cont'd)

For arbitrary wave packet with localized range of wave vectors, we may expand the dispersion equation about the central value $\mathbf{k}_0$:

$$\omega(\mathbf{k}) \approx \omega_0 + (\mathbf{k} - \mathbf{k}_0) \cdot \left(\frac{\partial \omega}{\partial \mathbf{k}} \right)_{\mathbf{k}_0} , \qquad \omega_0 \equiv \omega(\mathbf{k}_0) \,. \tag{56}$$

Inserting this approximation in the expression (51) for the wave packet gives

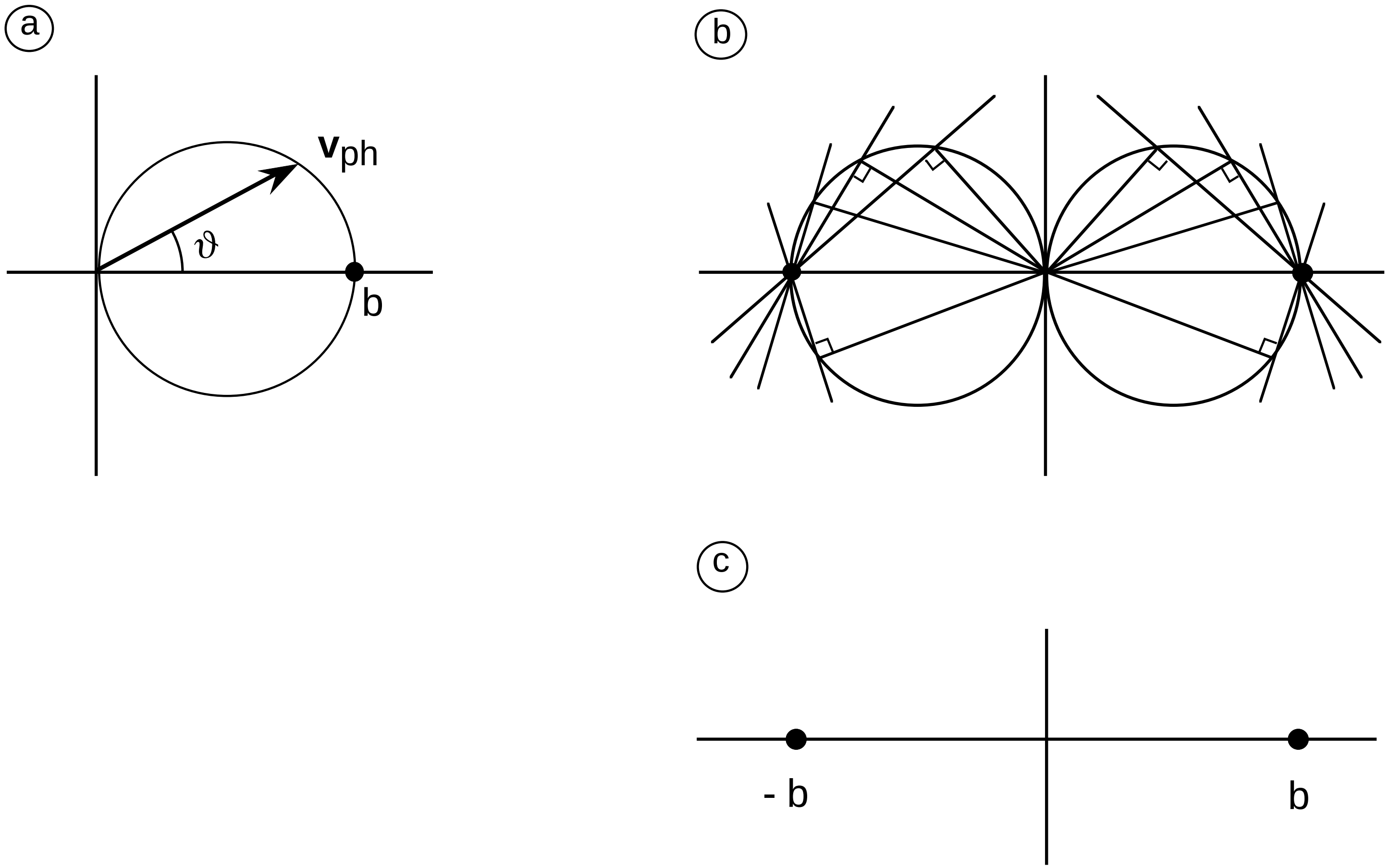
$$\Psi_i(\mathbf{r}, t) \approx \mathrm{e}^{\mathrm{i}(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)} \times \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} A_i(\mathbf{k}) \, \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}_0)\cdot(\mathbf{r} - (\partial\omega/\partial\mathbf{k})_{\mathbf{k}_0} t)} \, d^3k \,, \tag{57}$$

representing a carrier wave $\exp i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)$ with an amplitude-modulated envelope. Through constructive interference of the plane waves, *the envelope maintains its shape* during an extended interval of time, whereas *the surfaces of constant phase of the envelope move with the group velocity,*

$$\mathbf{v}_{\mathrm{gr}} = \left(\frac{d\mathbf{r}}{dt} \right)_{\mathrm{const.\,phase}} = \left(\frac{\partial \omega}{\partial \mathbf{k}} \right)_{\mathbf{k}_0} , \tag{58}$$

in agreement with the definition (48).

Example: Alfvén waves

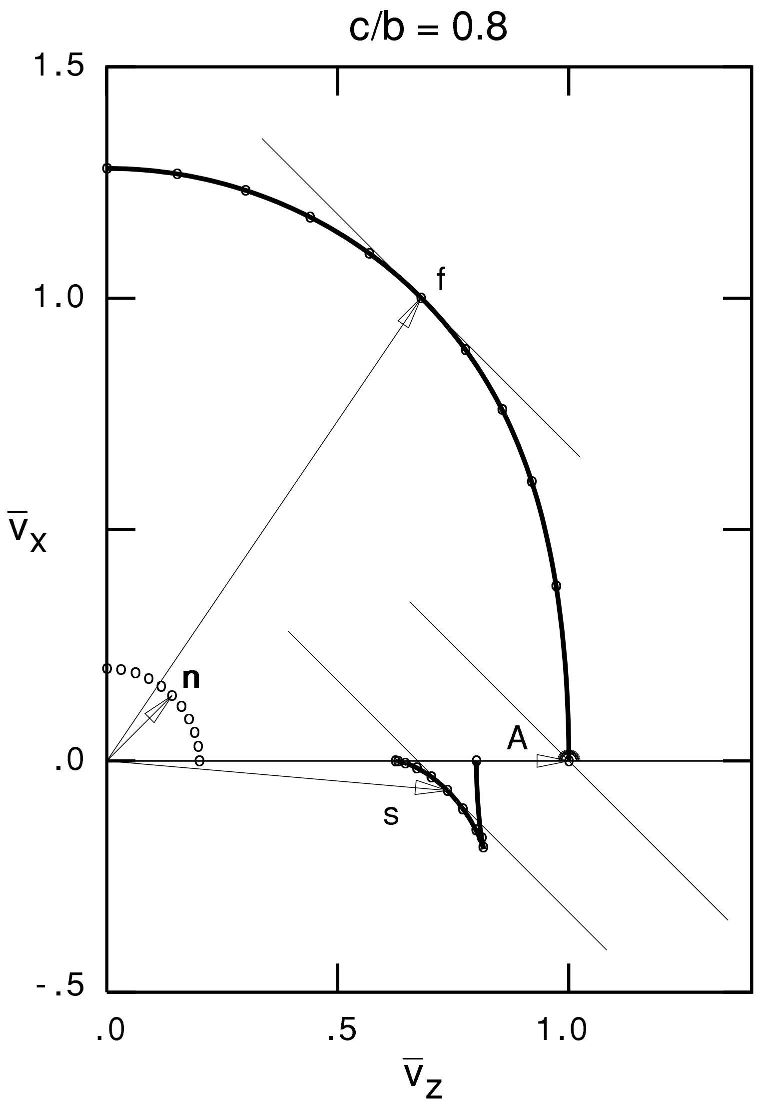


(a) Phase diagram for Alfvén waves is circle $\Rightarrow$ (b) wavefronts pass through points $\pm b$
$\Rightarrow$(c) those points are the group diagram.

Group diagram: queer behavior

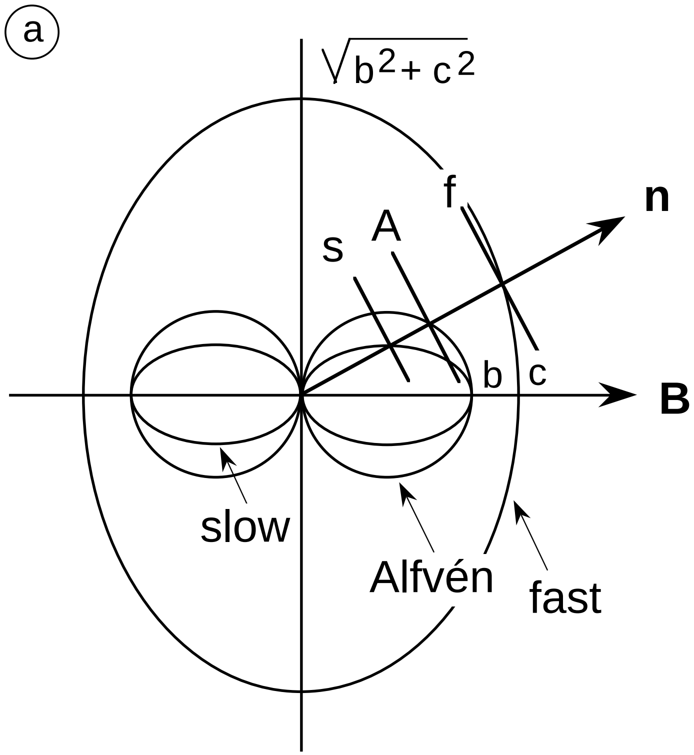
- *Group diagrams with $\mathbf{v}_{\rm gr}$ relative to $\mathbf{n}$ for the three MHD waves in the first quadrant.*

  Group velocities exhibit *mutually exclusive directions of propagation:* When $\mathbf{n}$ goes from $\vartheta = 0$ ($\parallel$ $\mathbf{B}$) to $\vartheta = \pi/2$ ($\perp$ $\mathbf{B}$), the fast group velocity changes from parallel to perpendicular (though it does not remain parallel to $\mathbf{n}$), the Alfvén group velocity remains purely parallel, but the slow group velocity initially changes *clockwise* from parallel to some negative angle and then back again to purely parallel. In the perpendicular direction, *slow wave packages propagate opposite to direction of* $\mathbf{n}$*!*

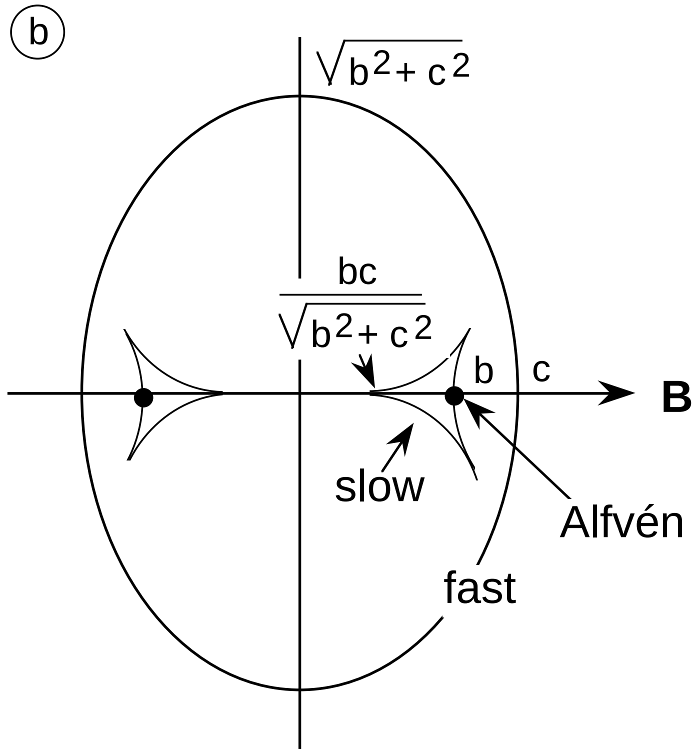


Friedrichs diagrams (schematic)

[exact diagrams in book: Fig. 5.5, parameter $c/b = \frac{1}{2}\gamma\beta\,,\ \beta \equiv 2p/B^2$]



Phase diagram
(plane waves)



Group diagram
(point disturbances)

Summary

- [*Entropy waves:* non-propagating density / entropy perturbations;]
- *Alfvén waves:* incompressible velocity perturbations $\perp$ plane of $\mathbf{k}$ & $\mathbf{B}$, preferably propagating $\parallel \mathbf{B}$;
- *Fast magnetoacoustic waves:* compressible velocity perturbations in the plane of $\mathbf{k}$ & $\mathbf{B}$, generalization of sound waves with contributions of the magnetic pressure, propagating in all directions but fastest $\perp \mathbf{B}$;
- *Slow magnetoacoustic waves:* compressible velocity perturbations in plane of $\mathbf{k}$ & $\mathbf{B}$, kind of sound waves with impeded propagation $\perp \mathbf{B}$ (orthogonal to fast modes).

Connection with next subject

Group diagram has a much wider applicability than just wave propagation in infinite homogeneous plasmas: Construction of wave packet involves contributions of large $\mathbf{k}$ (small wavelengths) so that the **concept of group velocity is essentially a local one**. It returns in *non-linear MHD of inhomogeneous plasmas*, where the associated concept of **characteristics** describes the propagation of initial data information through the plasma.

Example: point perturbation triggers MHD waves in uniform plasma (friedrichs.qt)

Method

- *Linear advection equation* in one spatial dimension with unknown $\Psi(x,t)$,

$$\frac{\partial \Psi}{\partial t} + u\,\frac{\partial \Psi}{\partial x} = 0\,, \tag{59}$$

  and given advection velocity u. For $u = \mathrm{const}$, the solution is trivial:

$$\Psi = f(x - ut)\,, \quad \text{where} \quad f = \Psi_0 \equiv \Psi(x, t=0)\,. \tag{60}$$

  $\Rightarrow$ Initial data Ψ_0 propagate along *characteristics:* parallel straight lines $dx/dt = u$.

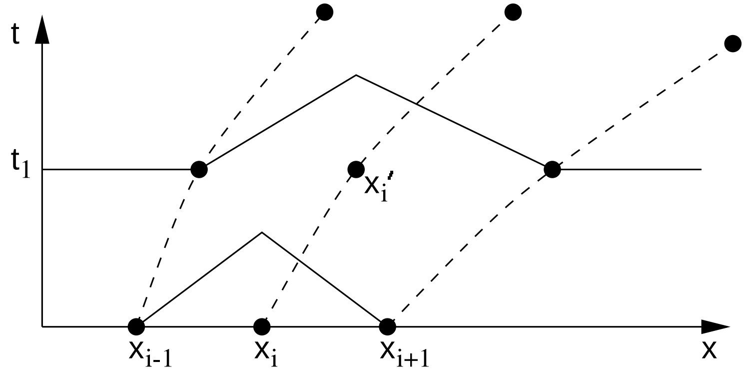
- For u not constant, characteristics become solutions of the ODEs

$$\frac{dx}{dt} = u(x,t)\,. \tag{61}$$

  Along these curves, solution $\Psi(x,t)$ of (59) is const:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} \equiv \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{dx}{dt} = 0\,. \tag{62}$$

  $\Rightarrow$ For given initial data, the solution can be determined at any time $t_1 > 0$ by constructing *characteristics through suitable set of points.* E.g., $\Psi(x_i', t_1) = \Psi_0(x_i)$ for 'tent' function.



Method (cont'd)

- The method of characteristics generalizes to *nonlinear* partial differential equations: basis of modern developments in *computational (magneto-)fluid dynamics* [C(M)FD].
- Example: *Quasi-linear advection equation* when u is also a function of the unknown Ψ itself. With $u = \Psi$, we obtain *Burgers' equation:*

$$\frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi}{\partial x} = \nu \frac{\partial^2 \Psi}{\partial x^2}, \tag{63}$$

  where viscous RHS models balance between nonlinear and dissipative processes. At first neglecting this small term, the characteristics are the solutions of the ODE

$$\frac{dx}{dt} = \Psi(x(t), t), \tag{64}$$

  which are just a set of straight lines with slopes determined by the initial data. For large times, the characteristics will cross, but the build-up of large gradients is counteracted by smoothing through the dissipative RHS term. This occurs in a very narrow region, so that a valid solution with a *shock* is obtained in the limit $\nu \to 0$.

Classification of PDEs

- Quasi-linear second order PDE in two dimensions:

$$A(\Phi_x, \Phi_y, x, y)\, \Phi_{xx} + 2B(\ldots)\, \Phi_{xy} + C(\ldots)\, \Phi_{yy} = D(\ldots)\,, \tag{65}$$

With $\Psi_1 \equiv \Phi_x\,, \Psi_2 \equiv \Phi_y \;\Rightarrow$ equivalent system of first order equations:

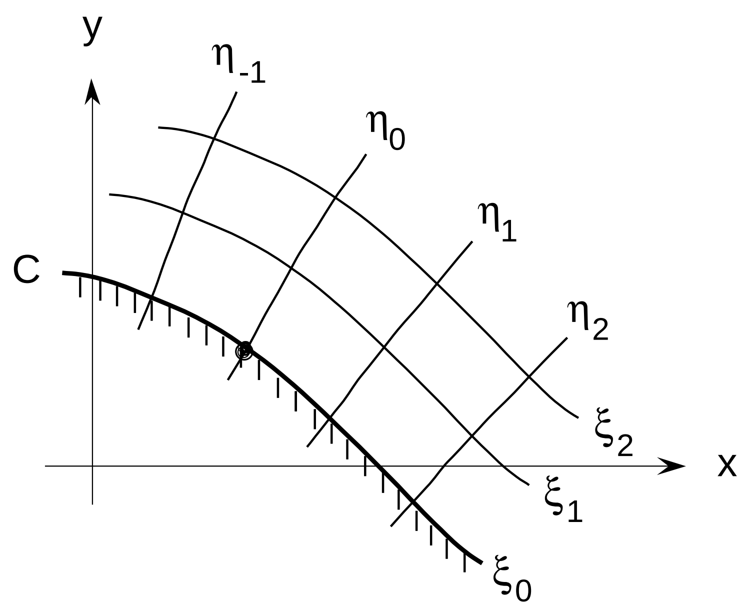
$$\begin{aligned} A(\Psi_1, \Psi_2, x, y)\, \Psi_{1x} + B(\ldots)\, \Psi_{1y} + B(\ldots)\, \Psi_{2x} + C(\ldots)\, \Psi_{2y} &= D(\ldots)\,, \\ \Psi_{1y} - \Psi_{2x} &= 0\,. \end{aligned} \tag{66}$$

Cauchy problem: find Ψ_1 and Ψ_2 away from boundary C when they are given on it.

- Replace coordinates x, y by boundary fitted coordinates ξ, η, where boundary C is given by $\xi(x, y) = \xi_0$. Boundary data become

$$\Psi_1(\xi_0, \eta) = f_1(\eta)\,, \quad \Psi_2(\xi_0, \eta) = f_2(\eta)\,. \tag{67}$$

We wish to investigate under which conditions $\Psi_1(\xi, \eta)$ and $\Psi_2(\xi, \eta)$ may be obtained by means of a power series solution about a particular point (ξ_0, η_0) on the boundary.



Classification (cont'd)

- Power series:

$$\begin{aligned} \Psi_1(\xi,\eta) &= \Psi_1(\xi_0,\eta_0) + (\xi-\xi_0)\left(\frac{\partial \Psi_1}{\partial \xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial \Psi_1}{\partial \eta}\right)_0 + \cdots , \\ \Psi_2(\xi,\eta) &= \Psi_2(\xi_0,\eta_0) + (\xi-\xi_0)\left(\frac{\partial \Psi_2}{\partial \xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial \Psi_2}{\partial \eta}\right)_0 + \cdots . \end{aligned} \quad (68)$$

  Green expressions known from boundary conditions (67) $\Rightarrow$ we need to investigate under which circumstances remaining expressions $(\partial \Psi_i / \partial \xi)_0$ can be calculated.

- Transform PDEs (66) to ξ-η coordinates:

$$\begin{aligned} &(A\xi_x + B\xi_y)\frac{\partial \Psi_1}{\partial \xi} + (B\xi_x + C\xi_y)\frac{\partial \Psi_2}{\partial \xi} \\ &\qquad = D - (A\eta_x + B\eta_y)\frac{\partial \Psi_1}{\partial \eta} - (B\eta_x + C\eta_y)\frac{\partial \Psi_2}{\partial \eta} , \quad (69) \\ &\xi_y \frac{\partial \Psi_1}{\partial \xi} - \xi_x \frac{\partial \Psi_2}{\partial \xi} = -\eta_y \frac{\partial \Psi_1}{\partial \eta} + \eta_x \frac{\partial \Psi_2}{\partial \eta} . \end{aligned}$$

  The unknown derivatives $\partial \Psi_1 / \partial \xi$ and $\partial \Psi_2 / \partial \xi$ may be determined from Eqs. (69) **if the determinant of the coefficients on the left hand side does not vanish**.

Classification (cont'd)

- Vice versa, condition that the determinant vanishes,

$$\begin{vmatrix} A\xi_x + B\xi_y & B\xi_x + C\xi_y \\ \xi_y & -\xi_x \end{vmatrix} = -A\xi_x^2 - 2B\xi_x\xi_y - C\xi_y^2 = 0\,, \qquad (70)$$

defines two directions in every point of the plane, *the characteristic directions*, **along which posing Cauchy boundary conditions does not determine the solution:**

$$\left.\frac{dy}{dx}\right|_{\text{char}} = -\frac{\xi_x}{\xi_y} = \frac{B \pm \sqrt{B^2 - AC}}{A}\,. \qquad (71)$$

- Three cases:

(a) $B^2 > AC \;\Rightarrow$ characteristics are real: *hyperbolic* equation (example: wave equation $\Phi_{xx} - (1/c^2)\Phi_{tt} = 0$);

(b) $B^2 = AC \;\Rightarrow$ characteristics are real but coincide: *parabolic* equation (example: heat equation $\Phi_{xx} - (1/\lambda)\Phi_t = 0$);

(c) $B^2 < AC \;\Rightarrow$ characteristics are complex: *elliptic* equation (example: Laplace's equation $\Phi_{xx} + \Phi_{yy} = 0$).

Apply to MHD equations

- Instead of 2-vector (Ψ_1, Ψ_2) : 8-vector $\Psi_i \ (i = 1, \cdots 8)$ for variables ρ, $\mathbf{v}$, e, $\mathbf{B}\ (\mathbf{r}, t)$.
- We will prove: MHD equations are *symmetric hyperbolic* PDEs; they posses complete set of *real characteristics related to the eigenvalues of the linearized system.*
- Apply same method as before: Assume boundary data for ρ, $\mathbf{v}$, e, $\mathbf{B}$ to be given on a 3-dimensional manifold in 4-dimensional space-time $\mathbf{r}, t$:

$$\xi(\mathbf{r}, t) = \xi_0 \,. \tag{72}$$

  (Visualize as being swept out by motion of 2-D surfaces in ordinary 3-D space $(\mathbf{r})$ when time t progresses.)

- **Duality**: – If this manifold is characteristic $\Rightarrow$ Cauchy problem ill-posed on it;
  – If this manifold is not characteristic $\Rightarrow$ Cauchy problem well-posed on it.
- Hence, for IVP in MHD (where $\rho(\mathbf{r}, 0)$, $\mathbf{v}(\mathbf{r}, 0)$, $e(\mathbf{r}, 0)$, $\mathbf{B}(\mathbf{r}, 0)$ are given on domain in ordinary 3-space) to be well-posed, ordinary 3-space should not be a characteristic.
- We will prove that the characteristics in MHD are real 3-dimensional manifolds involving time, so that *the IVP in MHD is well-posed.*

Application (cont'd)

- Cover 4-space $(\mathbf{r}, t)$ by boundary-fitted coordinates ξ, η, ζ, τ, and try power series:

$$\rho(\xi,\eta,\zeta,\tau) = \rho_0(\eta_0,\zeta_0,\tau_0) + (\xi-\xi_0)\left(\frac{\partial\rho}{\partial\xi}\right)_0 + (\eta-\eta_0)\left(\frac{\partial\rho}{\partial\eta}\right)_0 + (\zeta-\zeta_0)\left(\frac{\partial\rho}{\partial\zeta}\right)_0 + (\tau-\tau_0)\left(\frac{\partial\rho}{\partial\tau}\right)_0 + \cdots \quad \big(\text{etc. for } \mathbf{v},\, e,\, \mathbf{B}\big)\,. \tag{73}$$

- Problem solvable if unknowns $(\partial\rho/\partial\xi)_0$, $(\partial\mathbf{v}/\partial\xi)_0$, $(\partial e/\partial\xi)_0$, $(\partial\mathbf{B}/\partial\xi)_0$ can be constructed from MHD equations. Indicate those by a prime:

$$\begin{aligned}\nabla f &= \nabla\xi\, f' + \nabla\eta\,\frac{\partial f}{\partial\eta} + \nabla\zeta\,\frac{\partial f}{\partial\zeta} + \nabla\tau\,\frac{\partial f}{\partial\tau}\,,\\ \frac{\mathrm{D}f}{\mathrm{D}t} &= (\xi_t + \mathbf{v}\cdot\nabla\xi)\, f' + (\eta_t + \mathbf{v}\cdot\nabla\eta)\,\frac{\partial f}{\partial\eta} + (\zeta_t + \mathbf{v}\cdot\nabla\zeta)\,\frac{\partial f}{\partial\zeta} + (\tau_t + \mathbf{v}\cdot\nabla\tau)\,\frac{\partial f}{\partial\tau}.\end{aligned} \tag{74}$$

- Translation recipe (similar to shock recipe of Sec.4.5):

$$\begin{aligned}\nabla f &\to \mathbf{n} f' + \cdots\,, &\quad \mathbf{n} &\equiv \nabla\xi : \ \textit{normal to the characteristic},\\ \frac{\mathrm{D}f}{\mathrm{D}t} &\to -u f' + \cdots\,, &\quad -u &\equiv \xi_t + \mathbf{v}\cdot\nabla\xi : \ \textit{characteristic speed}.\end{aligned} \tag{75}$$

Application (cont'd)

- This gives:

$$
\begin{aligned}
-\,u\rho' + \rho\,\mathbf{n}\cdot\mathbf{v}' &= \cdots\,,\\
-\,\rho u\mathbf{v}' + (\gamma-1)\,\mathbf{n}\,(e\rho' + \rho e') + (\mathbf{n}\,\mathbf{B}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{B}' &= \cdots\,,\\
-\,ue' + (\gamma-1)e\,\mathbf{n}\cdot\mathbf{v}' &= \cdots\,,\\
-\,u\mathbf{B}' + (\mathbf{B}\,\mathbf{n}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{v}' &= \cdots\,, \qquad \mathbf{n}\cdot\mathbf{B}' = \cdots\,.
\end{aligned}
\tag{76}
$$

LHS analogous to EVP (30) for linear MHD waves, where $\mathbf{k}\rightarrow\mathbf{n}$ and $\omega\rightarrow u$!

- **Duality:** – Values $\rho'\,,\,\mathbf{v}'\,,\,e'\,,\,\mathbf{B}'$ may not be found if

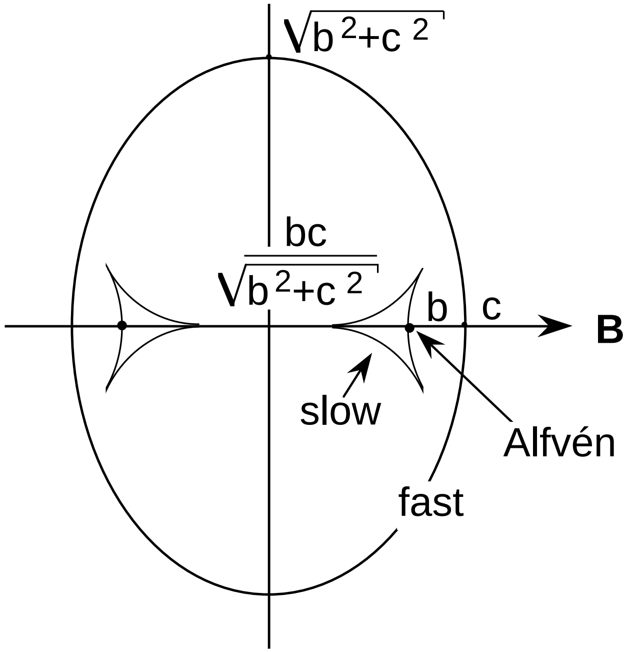
$$\Delta \equiv u(u^2 - b_n^2)\,\big[\,u^4 - (b^2+c^2)u^2 + b_n^2c^2\,\big] = 0 \quad\Rightarrow\quad \xi_0 \text{ characteristic}\,; \tag{77}$$

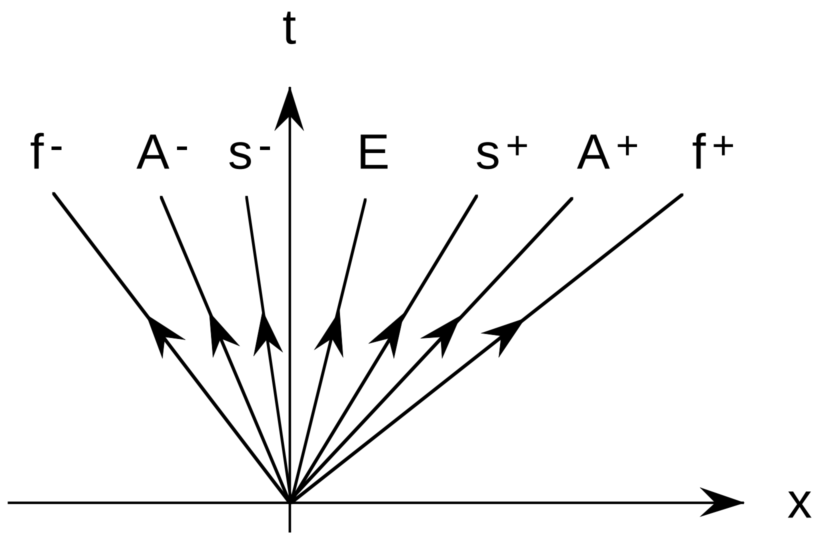
  – Values $\rho'\,,\,\mathbf{v}'\,,\,e'\,,\,\mathbf{B}'$ may be found if

$$\Delta \neq 0 \quad\Rightarrow\quad \xi_0 \text{ not characteristic (solutions may be propagated away from it)}\,.$$

$\Rightarrow$ *7 real characteristics*, corresponding to 7 linear waves (entropy, Alfvén, slow, fast). **The equations of ideal MHD are symmetric hyperbolic equations**, and **the initial value problem is well-posed** (Friedrichs).

Application (cont'd)





Group diagram is the *ray surface*, i.e. the spatial part of characteristic manifold at certain time t_0.

x-t cross-sections of 7 characteristics (x-axis oblique with respect to $\mathbf{B}$; inclination of entropy mode E indicates plasma background flow).

- *Locality* of group diagrams and characteristics neglects *global plasma inhomogeneity.* $\Rightarrow$ Next topic is *waves and instabilities in inhomogeneous plasmas*.