Flow: Waves and instabilities in stationary plasmas

Overview

- **Introduction:** theoretical themes for a complete MHD description of laboratory and astrophysical plasmas, static versus stationary plasmas;
- **Spectral theory of stationary plasmas:** Frieman–Rotenberg formalism for waves and instabilities, quadratic eigenvalue problem gives complex eigenvalues, implications of the Doppler shift for the continuous spectra;
- **Kelvin–Helmholtz instability of streaming plasmas:** gravitating plasma with an interface where the velocity changes discontinuously, influence of the magnetic field;
- **Magneto-rotational instability of rotating plasmas:** derivation of the dispersion equation, growth rates of instabilities, application to accretion disks.

Theoretical themes

- In overview of magnetic structures and dynamics (Chap. 8), we encountered:
  - Central importance of magnetic flux tubes $\Rightarrow$ **Cylindrical plasmas**, 1D: $f(r)$
    [Volume 1: Chap. 9]
  - Astrophysical flows (winds, disks, jets) $\Rightarrow$ **Plasmas with background flow**
    [this lecture, Volume 2: MHDF.pdf]
  - Explosive phenomena due to reconnection $\Rightarrow$ **Resistive MHD**
    [Volume 2: MHDR.pdf]
  - Magnetic confinement for fusion (tokamak) $\Rightarrow$ **Toroidal plasmas**, 2D: $f(r, \vartheta)$
    [Volume 2: MHDT.pdf]
  - Shocks, transonic flows, dynamos, turbulence $\Rightarrow$ **Nonlinear MHD**
    [Volume 2: MHDS.pdf]
  - All plasma dynamics (e.g. space weather) $\Rightarrow$ **Computational MHD**
    [Volume 2: . . .]
- **MHD with background flow** is the most urgent topic (also for fusion research since divertors and neutral beam injection cause significant flows in tokamaks).

  $\Rightarrow$ **From static ($\mathbf{v} = 0$) to stationary ($\mathbf{v} \neq 0$) plasmas!**

Static versus stationary plasmas

- Starting point is the *set of nonlinear ideal MHD equations:*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}) + \nabla p - \mathbf{j}\times\mathbf{B} - \rho\mathbf{g} = 0\,, \qquad \mathbf{j} = \nabla\times\mathbf{B}\,, \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p \nabla\cdot\mathbf{v} = 0\,, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B}) = 0\,, \qquad \nabla\cdot\mathbf{B} = 0\,, \tag{4}$$

with gravitational acceleration $\mathbf{g} = -\nabla\Phi_{\rm gr}$ due to external gravity field $\Phi_{\rm gr}$.

- Recall the simplicity of **static equilibria** ($\partial/\partial t = 0\,,\mathbf{v} = 0$),

$$\nabla p = \mathbf{j}\times\mathbf{B} + \rho\mathbf{g}\,, \qquad \mathbf{j} = \nabla\times\mathbf{B}\,, \qquad \nabla\cdot\mathbf{B} = 0\,, \tag{5}$$

with perturbations described by **self-adjoint operator $\mathbf{F}$ with real eigenvalues ω^2:**

$$\mathbf{F}(\boldsymbol{\xi}) = \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} \quad \Rightarrow \quad \mathbf{F}(\hat{\boldsymbol{\xi}}) = -\rho\omega^2\hat{\boldsymbol{\xi}}\,. \tag{6}$$

- **Can one construct a similar powerful scheme for stationary plasmas ($\mathbf{v} \neq 0$)?**

Stationary equilibria

- Basic nonlinear ideal MHD equations for **stationary equilibria** ($\partial/\partial t = 0$):

$$\nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{7}$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} \,, \qquad \mathbf{j} = \nabla \times \mathbf{B} \,, \tag{8}$$

$$\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \,, \tag{9}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,. \tag{10}$$

$\Rightarrow$ None of them trivially satisfied now (except for simple geometries)!

- For *plane gravitating plasma slab,* equilibrium unchanged w.r.t. static case:

$$(p + \tfrac{1}{2}B^2)' = -\rho g \qquad ({}' \equiv d/dx\,) \,. \tag{11}$$

- For *cylindrical plasma,* the equilibrium is changed significantly by the *centrifugal acceleration,* $-\mathbf{v} \cdot \nabla \mathbf{v} = (v_\theta^2/r)\mathbf{e}_r$:

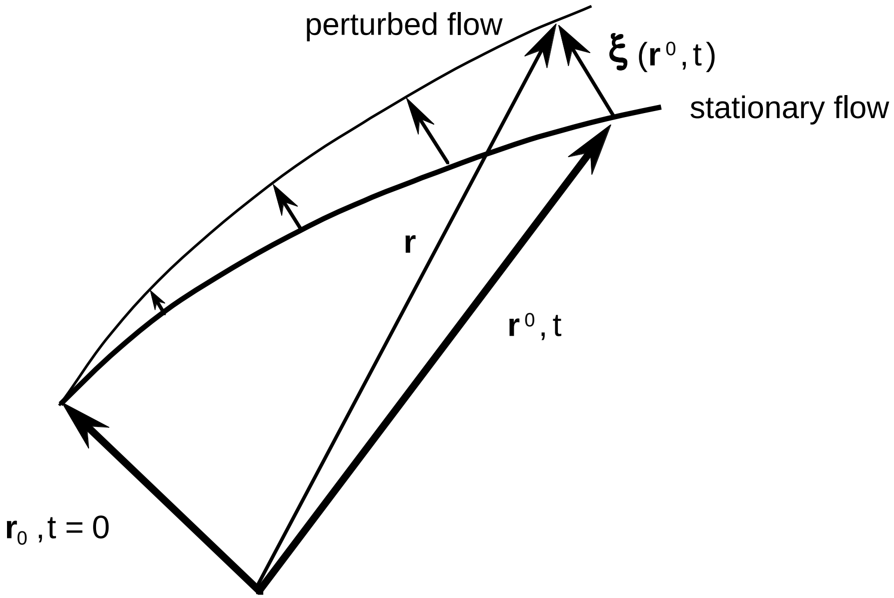
$$(p + \tfrac{1}{2}B^2)' + \frac{1}{r}B_\theta^2 = \frac{1}{r}\rho v_\theta^2 - \rho \Phi_{\rm gr}' \qquad ({}' \equiv d/dr\,) \,. \tag{12}$$

$\Rightarrow$ Modifications for plane and cylindrical stationary flows quite different:
translations and rotations are physically different phenomena.

Frieman–Rotenberg formalism

[Rev. Mod. Phys. **32**, 898 (1960)]

Spectral theory for general stationary equilibria (no further simplifying assumptions):



- First, construct **displacement vector ξ** connecting perturbed flow at position $\mathbf{r}$ with unperturbed flow at position $\mathbf{r}^0$:

$$\mathbf{r}(\mathbf{r}^0, t) = \mathbf{r}^0 + \boldsymbol{\xi}(\mathbf{r}^0, t)\,. \qquad (13)$$

- **In terms of the coordinates $(\mathbf{r}^0, t)$, the equilibrium is time-independent:**

  $\rho = \rho^0(\mathbf{r}^0)$, etc., satisfiying (7)–(10).

- Gradient

$$\nabla = (\nabla \mathbf{r}^0) \cdot \nabla^0 = \nabla(\mathbf{r} - \boldsymbol{\xi}) \cdot \nabla^0 \approx \nabla^0 - (\nabla^0 \boldsymbol{\xi}) \cdot \nabla^0\,, \qquad (14)$$

  and Lagrangian time derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t}\Big|_{\mathbf{r}^0} + \mathbf{v}^0 \cdot \nabla^0\,, \qquad (15)$$

  yield expression for the velocity at the perturbed trajectory:

$$\mathbf{v}(\mathbf{r}^0 + \boldsymbol{\xi}, t) \equiv \frac{\mathrm{D}\mathbf{r}}{\mathrm{D}t} = \frac{\mathrm{D}\mathbf{r}^0}{\mathrm{D}t} + \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \mathbf{v}^0 + \mathbf{v}^0 \cdot \nabla^0 \boldsymbol{\xi} + \frac{\partial \boldsymbol{\xi}}{\partial t}\,. \qquad (16)$$

Frieman–Rotenberg formalism (cont'd)

- Linearization of Eqs. (1), (3), (4) gives perturbed quantities in terms of $\boldsymbol{\xi}$ alone:

$$\rho \approx \rho^0 - \rho^0 \nabla^0 \cdot \boldsymbol{\xi} \qquad = \rho^0 + \rho^1_{\rm E} + \boldsymbol{\xi} \cdot \nabla^0 \rho^0 \,, \tag{17}$$

$$p \approx p^0 - \gamma p^0 \nabla^0 \cdot \boldsymbol{\xi} \qquad = p^0 + \pi + \boldsymbol{\xi} \cdot \nabla^0 p^0 \,, \qquad (\, \pi \equiv p^1_{\rm E} \,) \tag{18}$$

$$\mathbf{B} \approx \mathbf{B}^0 + \mathbf{B}^0 \cdot \nabla^0 \boldsymbol{\xi} - \mathbf{B}^0 \nabla^0 \cdot \boldsymbol{\xi} \quad = \mathbf{B}^0 + \mathbf{Q} + \boldsymbol{\xi} \cdot \nabla^0 \mathbf{B}^0 \,, \quad (\, \mathbf{Q} \equiv \mathbf{B}^1_{\rm E} \,) \tag{19}$$

where we will now drop the superscripts ^{0} (since everything has this superscript).

- Substitution in Eq. (2) ($+$ algebra!) $\Rightarrow$ **spectral equation for equilibria with flow:**

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\rho \mathbf{v} \cdot \nabla \frac{\partial \boldsymbol{\xi}}{\partial t} - \mathbf{G}(\boldsymbol{\xi}) = 0 \,, \tag{20}$$

$$\mathbf{G} \equiv \mathbf{F} + \nabla \cdot (\boldsymbol{\xi}\, \rho \mathbf{v} \cdot \nabla \mathbf{v} - \rho \mathbf{v} \mathbf{v} \cdot \nabla \boldsymbol{\xi}) \,, \tag{21}$$

$$\mathbf{F} \equiv -\nabla \pi - \mathbf{B} \times (\nabla \times \mathbf{Q}) + (\nabla \times \mathbf{B}) \times \mathbf{Q} + (\nabla \Phi)\, \nabla \cdot (\rho \boldsymbol{\xi}) \,.$$

- For normal modes, $\boldsymbol{\xi} \sim \exp(-\mathrm{i}\omega t)$, a quadratic eigenvalue problem is obtained:

$$\mathbf{G}(\boldsymbol{\xi}) + 2\mathrm{i}\rho\omega \mathbf{v} \cdot \nabla \boldsymbol{\xi} + \rho \omega^2 \boldsymbol{\xi} = 0 \,, \tag{22}$$

where the **generalised force operator $\mathbf{G}$ is selfadjoint (like $\mathbf{F}$ in the static case) but eigenvalues ω are complex because of the Doppler shift operator $\mathrm{i}\mathbf{v} \cdot \nabla$.**

Spectral equation for plane slab

- For the plane slab model of Chapter 7, extended with a *plane shear flow field,*

$$\mathbf{B} = B_y(x)\mathbf{e}_y + B_z(x)\mathbf{e}_z\,, \quad \rho = \rho(x)\,, \quad p = p(x)\,,$$
$$\mathbf{v} = v_y(x)\mathbf{e}_y + v_z(x)\mathbf{e}_z\,, \qquad (23)$$

the equilibrium is unchanged and the two new terms in $\mathbf{G}$ yield: $\mathbf{v}\cdot\nabla\mathbf{v} = 0$ and $-\nabla\cdot(\rho\mathbf{v}\mathbf{v}\cdot\nabla\boldsymbol{\xi}) = -\rho(\mathbf{v}\cdot\nabla)^2\boldsymbol{\xi}$, so that the eigenvalue problem (22) becomes:

$$\mathbf{G}(\boldsymbol{\xi}) + 2\mathrm{i}\rho\omega\mathbf{v}\cdot\nabla\boldsymbol{\xi} + \rho\omega^2\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) + \rho(\omega + \mathrm{i}\mathbf{v}\cdot\nabla)^2 = 0$$

$$\Rightarrow \quad \mathbf{F}(\boldsymbol{\xi}) = -\rho\widetilde{\omega}^2\,\boldsymbol{\xi}\,, \qquad \widetilde{\omega} \equiv \omega + \mathrm{i}\mathbf{v}\cdot\nabla\,. \qquad (24)$$

- Hence, the equations for the static slab remain valid with the replacement

$$\omega \quad \to \quad \widetilde{\omega}(x) \equiv \omega - \Omega_0(x)\,, \quad \Omega_0 \equiv \mathbf{k}_0\cdot\mathbf{v}(x)\,, \qquad (25)$$

where $\Omega_0(x)$ is the local Doppler shift and $\widetilde{\omega}(x)$ is the *local Doppler shifted frequency* observed in a local frame co-moving with the plasma at the vertical position x.

- Since $\widetilde{\omega}$ depends on x, eigenvalues will be shifted by some average of $\Omega_0(x)$ across the layer. If a static equilibrium is unstable (eigenvalue on positive imaginary axis), for the corresponding equilibrium with flow that eigenvalue moves into the complex plane and becomes an *overstable mode.*

HD continua for plane shear flow

- How are the MHD waves affected by background flow of the plasma?

  First, consider **continuous spectrum in the HD case** for plane slab geometry, inhomogeneous fluid with horizontal flow:

$$\mathbf{v} = v_y(x)\mathbf{e}_y + v_z(x)\mathbf{e}_z \,.$$

- Lagrangian time derivative:

$$(\mathrm{D}f/\mathrm{d}t)_1 \equiv (\partial f/\partial t + \mathbf{v}\cdot\nabla f)_1 = -\mathrm{i}\,\widetilde{\omega} f_1 + f_0{}' v_{1x} \,, \qquad \widetilde{\omega} \equiv \omega - \mathbf{k_0}\cdot\mathbf{v} \,,$$

  $\widetilde{\omega}(x)$: frequency observed in local frame co-moving with fluid layer at position x.

- Singularities when $\widetilde{\omega} = 0$ somewhere in the fluid $\Rightarrow$ **HD flow continuum $\{\Omega_0(x)\}$,** consisting of the zeros of the local Doppler shifted frequency

$$\widetilde{\omega} \equiv \omega - \Omega_0(x) \,, \quad \Omega_0 \equiv -\mathrm{i}\,\mathbf{v}\cdot\nabla = \mathbf{k_0}\cdot\mathbf{v} \,, \quad \text{on the interval } x_1 \le x \le x_2 \,.$$

  These have been extensively investigated in the hydrodynamics literature.

  [Lin 1955; Case 1960; Drazin and Reid, Hydrodynamic Stability (Cambridge, 2004)]

MHD continua for plane shear flow

- **Forward (+) / backward (−) Alfvén and slow continua, and fast cluster points:**

$$\Omega_A^\pm \equiv \Omega_0 \pm \omega_A\,, \qquad \omega_A \equiv F/\sqrt{\rho}\,, \qquad F \equiv -\mathrm{i}\,\mathbf{B}\cdot\nabla = \mathbf{k_0}\cdot\mathbf{B}\,,$$

$$\Omega_S^\pm \equiv \Omega_0 \pm \omega_S\,, \qquad \omega_S \equiv \sqrt{\frac{\gamma p}{\gamma p + B^2}}\,F/\sqrt{\rho}\,, \qquad \Omega_0 \equiv -\mathrm{i}\,\mathbf{v}\cdot\nabla = \mathbf{k_0}\cdot\mathbf{v}\,,$$

$$\Omega_F^\pm \equiv \pm\infty\,.$$

- The flow contribution to the MHD continua creates the following ordering of the local frequencies **(which are all real)** in the co-moving frame:

$$\Omega_F^- \le \Omega_{f0}^- \le \Omega_A^- \le \Omega_{s0}^- \le \Omega_S^- \le \Omega_0 \le \Omega_S^+ \le \Omega_{s0}^+ \le \Omega_A^+ \le \Omega_{f0}^+ \le \Omega_F^+\,.$$

- **The discrete spectra are monotonic for real ω outside these frequencies.**

- In the limit $\mathbf{B} \to 0$, the Alfvén and slow continua collapse into the flow continuum,

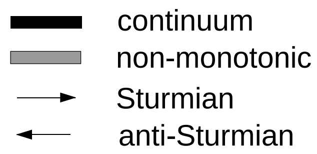
$$\Omega_A^\pm \to \Omega_0\,, \qquad \Omega_S^\pm \to \Omega_0 \qquad (\text{whereas } \Omega_F^\pm \text{ remains at } \pm\infty)\,.$$

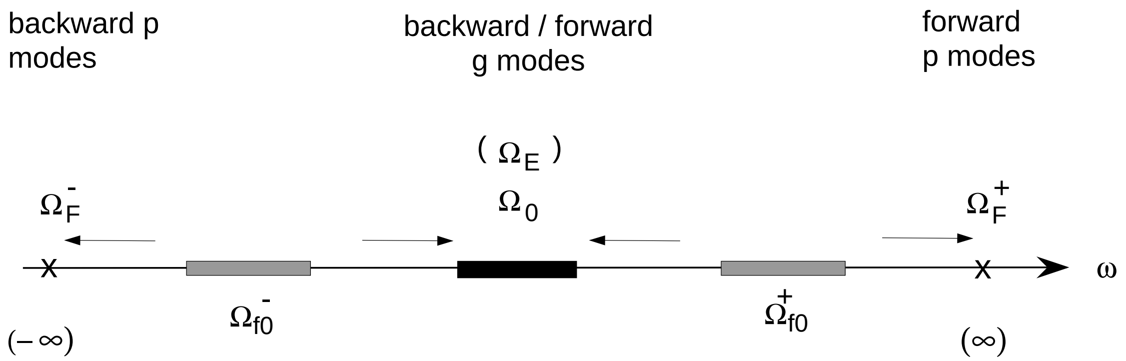
  Vice versa, the HD flow continuum is absorbed by the MHD continua when $B \neq 0$. Hence, contrary to the literature, **there is no separate flow continuum in MHD!**

  [Goedbloed, Beliën, van der Holst, Keppens, Phys. Plasmas **11**, 4332 (2004)]

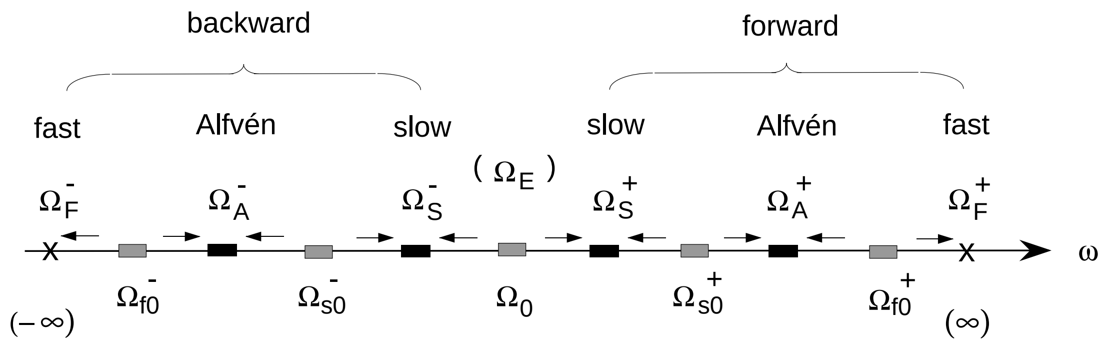
Real parts of HD & MHD spectra

- **HD spectrum** of fluid flow:





- **MHD spectrum** of plasma flow:



Kelvin–Helmholtz instability: equilibrium

- Extend Rayleigh–Taylor instability of plasma–vacuum interface (sheets 6-36 – 6-42) to **plasma–plasma interface with two different velocities** (see figure on 6-36): *Rayleigh–Taylor + Kelvin–Helmholtz!*

- Upper layer ($0 < x \leq a$):

$$\rho = \mathrm{const}\,, \quad \mathbf{v} = (0, v_y, v_z) = \mathrm{const}\,, \quad \mathbf{B} = (0, B_y, B_z) = \mathrm{const}\,,$$
$$p' = -\rho g \qquad \Rightarrow \quad p = p_0 - \rho g x \quad (p_0 \geq \rho g a)\,. \tag{26}$$

  Lower layer ($-b \leq x < 0$):

$$\hat\rho = \mathrm{const}\,, \quad \hat{\mathbf{v}} = (0, \hat v_y, \hat v_z) = \mathrm{const}\,, \quad \hat{\mathbf{B}} = (0, \hat B_y, \hat B_z) = \mathrm{const}\,,$$
$$\hat p' = -\hat\rho g \qquad \Rightarrow \quad \hat p = \hat p_0 - \hat\rho g x\,. \tag{27}$$

- Jumps at the interface ($x = 0$):

$$\text{BC:} \quad p_0 + \tfrac{1}{2}B_0^2 = \hat p_0 + \tfrac{1}{2}\hat B_0^2 \qquad \text{(pressure balance)}\,, \tag{28}$$

$$\Rightarrow \quad \mathbf{j}^\star = \mathbf{n} \times [\![\mathbf{B}]\!] = \mathbf{e}_x \times (\mathbf{B} - \hat{\mathbf{B}}) \qquad \text{(surface current)}\,,$$
$$\Rightarrow \quad \boldsymbol{\omega}^\star = \mathbf{n} \times [\![\mathbf{v}]\!] = \mathbf{e}_x \times (\mathbf{v} - \hat{\mathbf{v}}) \qquad \text{(surface vorticity)}\,. \tag{29}$$

Kelvin–Helmholtz instability: normal modes

- Now (in contrast to energy principle analysis of 6-36 – 6-42), **normal mode analysis:**

$$\boldsymbol{\xi} \sim \exp\left[\mathrm{i}(k_y y + k_z z - \omega t)\right].$$

- For *incompressible plasma,* taking limit $c^2 \equiv \gamma p/\rho \to \infty$ of Eq. (50) on sheet 7-20, *with plane flow,* replacing $\omega \to \widetilde{\omega}$ (Eq. (25) of sheet F-7), basic ODE becomes:

$$\frac{d}{dx}\left[\rho(\widetilde{\omega}^2 - \omega_A^2)\frac{d\xi}{dx}\right] - k_0^2\left[\rho(\widetilde{\omega}^2 - \omega_A^2) + \rho' g\right]\xi = 0. \qquad (30)$$

Doppler shifted freq. $\widetilde{\omega} \equiv \omega - \Omega_0\,,\ \Omega_0 \equiv \mathbf{k}_0 \cdot \mathbf{v}$; Alfvén freq. $\omega_A \equiv \mathbf{k}_0 \cdot \mathbf{B}/\sqrt{\rho_0}$.

- In this case, all equilibrium quantities constant so that ODEs simplify to equations with constant coefficients:

$$\xi'' - k_0^2 \xi = 0\,,\quad \text{BC}\ \ \xi(a) = 0 \quad \Rightarrow \quad \xi = C \sinh\left[k_0(a - x)\right], \qquad (31)$$

$$\hat{\xi}'' - k_0^2 \hat{\xi} = 0\,,\quad \text{BC}\ \ \hat{\xi}(-b) = 0 \quad \Rightarrow \quad \hat{\xi} = \hat{C} \sinh\left[k_0(x + b)\right]. \qquad (32)$$

$\Rightarrow$ *Surface modes* (cusp-shaped eigenfunctions). This part is trivial, contains hardly any physics. Physics comes from the *BCs at $x = 0$ determining the eigenvalues.*

Kelvin–Helmholtz instability: interface conditions

- Now (in contrast to energy principle analysis of 6-36 – 6-42), *need both interface conditions* (model II* BCs), to determine relative amplitude $\hat{C}/C$ and eigenvalue ω:

  – *First interface condition* (continuity of normal velocity):

$$[\![\mathbf{n}\cdot\boldsymbol{\xi}]\!] = 0 \quad \Rightarrow \quad \xi(0) = \hat{\xi}(0) = 0 \quad \Rightarrow \quad C \sinh\, k_0 a = \hat{C} \sinh\, k_0 b\,. \tag{33}$$

  – *Second interface condition* (pressure balance):

$$[\![\,\Pi + \mathbf{n}\cdot\boldsymbol{\xi}\,\mathbf{n}\cdot\nabla(p + \tfrac{1}{2}B^2)\,]\!] = 0\,, \quad \Pi \equiv -\gamma p \nabla\cdot\boldsymbol{\xi} - \xi\cdot\nabla p + \mathbf{B}\cdot\mathbf{Q}\,, \tag{34}$$

  where $\gamma p \nabla\cdot\boldsymbol{\xi}$ is undetermined. Determine Π from expression for compressible plasmas, Book, Eq. (7.99), with ω replaced by $\widetilde{\omega}$ and taking limit $\gamma \to \infty$:

$$\Pi \equiv -\frac{N}{D}\,\xi' + \rho g\, \frac{\widetilde{\omega}^2(\widetilde{\omega}^2 - \omega_A^2)}{D}\,\xi \;\to\; \frac{\rho}{k_0^2}(\widetilde{\omega}^2 - \omega_A^2)\xi'\,. \tag{35}$$

- Dividing the second by the first interface condition then gives

$$\left[\!\!\left[\, \frac{\rho}{k_0^2}(\widetilde{\omega}^2 - \omega_A^2)\frac{\xi'}{\xi} - \rho g \,\right]\!\!\right] = 0 \quad \Rightarrow \quad \text{eigenvalue } \omega\,. \tag{36}$$

Kelvin–Helmholtz instability: dispersion equation

- Inserting solutions (31) and (32) for ξ and $\hat{\xi}$ yields the **dispersion equation:**

$$-\rho\left[(\omega-\Omega_0)^2-\omega_A^2\right]\coth(k_0a)-k_0\rho g=\hat{\rho}\left[(\omega-\hat{\Omega}_0)^2-\hat{\omega}_A^2\right]\coth(k_0b)-k_0\hat{\rho}g\,. \qquad (37)$$

Describes magnetic field line bending (Alfvén), gravity (RT), velocity difference (KH).

- Approximations for long wavelengths ($k_0x \ll 1$): $\coth k_0x \approx (k_0x)^{-1}$,
  short wavelengths ($k_0x \gg 1$): $\coth k_0x \approx 1$.

- Solutions for short wavelengths (walls effectively at ∞ and $-\infty$):

$$\omega=\frac{\rho\Omega_0+\hat{\rho}\hat{\Omega}_0}{\rho+\hat{\rho}}\pm\sqrt{-\frac{\rho\hat{\rho}(\Omega_0-\hat{\Omega}_0)^2}{(\rho+\hat{\rho})^2}+\frac{\rho\omega_A^2+\hat{\rho}\hat{\omega}_A^2}{\rho+\hat{\rho}}-\frac{k_0(\rho-\hat{\rho})g}{\rho+\hat{\rho}}}\,. \qquad (38)$$

$\Rightarrow$ Stable (square root real) if

$$(\mathbf{k}_0\cdot\mathbf{B})^2+(\mathbf{k}_0\cdot\hat{\mathbf{B}})^2>\frac{\rho\hat{\rho}}{\rho+\hat{\rho}}\left[\mathbf{k}_0\cdot(\mathbf{v}-\hat{\mathbf{v}})\right]^2+k_0(\rho-\hat{\rho})g\,. \qquad (39)$$

magnetic shear | K–H drive | R–T drive

Kelvin–Helmholtz instability: generic transitions

- Pure KH instability ($\mathbf{B} = \hat{\mathbf{B}} = 0$, $g = 0$, $\mathbf{k}_0 \parallel \mathbf{v} \parallel \hat{\mathbf{v}}$):

$$\omega = k_0 \left[\frac{\rho v + \hat{\rho}\hat{v}}{\rho + \hat{\rho}} \pm \mathrm{i} \frac{\sqrt{\rho\hat{\rho}}}{\rho + \hat{\rho}} |v - \hat{v}| \right] . \qquad (40)$$

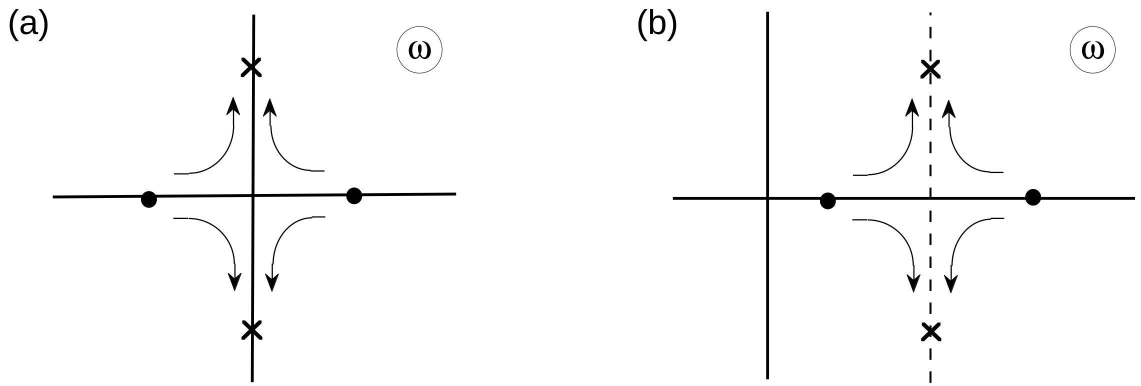
$\Rightarrow$ Degeneracy of Doppler mode $\omega = k_0 v$ lifted by $v \neq \hat{v}$.

- Doppler shifted RT instability ($\mathbf{B} = \hat{\mathbf{B}} = 0$, $\mathbf{v} = \hat{\mathbf{v}}$, $\mathbf{k}_0 \parallel \mathbf{v}$):

$$\omega = k_0 v \pm \mathrm{i} \sqrt{\frac{k_0(\rho - \hat{\rho})g}{\rho + \hat{\rho}}} . \qquad (41)$$

$\Rightarrow$ Degeneracy of Doppler mode $\omega = k_0 v$ lifted by $\rho \neq \hat{\rho}$.

- Hence, generic *transitions to instability* for (a) static, and (b) stationary plasmas:



Exp. growth: through origin *Overstability: through real axis*

Kelvin–Helmholtz instability: generalizations

- Of course, the assumption of two homogeneous plasma layers with a velocity difference at the interface (made to make the analysis tractable for a relevant instability) evades the basic problems of **diffuse plasma flows:** continuous spectra, cluster points, and *eigenvalues on unknown paths in the complex ω plane.*

  $\Rightarrow$ Further progress only by *linear computational methods:* finite differences and finite elements, spectral methods, linear system solvers, etc.

- Instabilities always grow towards amplitudes that necessitate consideration of the **nonlinear evolution**: *coupling of linear modes, nonlinear saturation, and turbulence* appear: see simulation of Rayleigh–Taylor instability with Versatile Advection Code, where secondary Kelvin–Helmholtz instabilities develop (sheet 6-42).

  $\Rightarrow$ Further progress mainly by *nonlinear computational methods:* implicit and semi-implicit time stepping, finite volume methods, shock-capturing methods, etc.

Recap.: MHD wave equation in cylinder (static) [Vol. 1: Chap. 9]

- *Fourier harmonics* $\hat{\boldsymbol{\xi}}(r; m, k) \exp\left[\mathrm{i}(m\theta + kz)\right]$, keep differential operators d/dr.
- *Field line projection* in normal, perpendicular, and parallel directions:

$$\mathbf{e}_r \equiv \nabla r\,, \qquad \mathbf{e}_\perp(r) \equiv (\mathbf{B}/B) \times \mathbf{e}_r\,, \quad \mathbf{e}_\parallel(r) \equiv \mathbf{B}/B\,, \tag{42}$$

$$\partial_r \equiv d/dr\,,$$

$$\nabla = \mathbf{e}_r \partial_r + \mathbf{e}_\theta r^{-1} \partial_\theta + \mathbf{e}_z \partial_z \quad \Rightarrow \quad G(r) \equiv mB_z/r - kB_\theta\,, \tag{43}$$

$$\big(\text{watch out: } \partial_\theta \mathbf{e}_r = \mathbf{e}_\theta\,, \;\; \partial_\theta \mathbf{e}_\theta = -\mathbf{e}_r!\big) \qquad F(r) \equiv mB_\theta/r + kB_z\,,$$

$$\boldsymbol{\xi} = \xi\,\mathbf{e}_r - \mathrm{i}\eta\,\mathbf{e}_\perp - \mathrm{i}\zeta\,\mathbf{e}_\parallel\,. \tag{44}$$

MHD spectral equation $\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2 \boldsymbol{\xi}$ (+ algebra!) $\Rightarrow$ *Vector eigenvalue problem:*

$$\begin{pmatrix}
\dfrac{d}{dr}\dfrac{\gamma p + B^2}{r}\dfrac{d}{dr} r - F^2 - r\left(\dfrac{B_\theta^2}{r^2}\right)' & \dfrac{d}{dr}\dfrac{G}{B}(\gamma p + B^2) - 2\dfrac{kB_\theta B}{r} & \dfrac{d}{dr}\dfrac{F}{B}\gamma p \\
-\dfrac{G}{B}\dfrac{\gamma p + B^2}{r}\dfrac{d}{dr} r - 2\dfrac{kB_\theta B}{r} & -\dfrac{G^2}{B^2}(\gamma p + B^2) - F^2 & -\dfrac{FG}{B^2}\gamma p \\
-\dfrac{F}{B}\dfrac{\gamma p}{r}\dfrac{d}{dr} r & -\dfrac{FG}{B^2}\gamma p & -\dfrac{F^2}{B^2}\gamma p
\end{pmatrix}
\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}
= -\rho\omega^2 \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{45}$$

- Eliminate perpendicular and parallel components η and ζ :

$$\eta = \frac{\rho(\gamma p + B^2) G (\omega^2 - \omega_S^2)\, r\chi' + 2kB_\theta (B^2\rho\omega^2 - \gamma p F^2)\,\chi}{r^2 BD}\,, \qquad (46)$$

$$\zeta = \frac{\rho\gamma p F[(\omega^2 - \omega_A^2)\, r\chi' + 2kB_\theta G\,\chi]}{r^2 BD}\,,$$

- Substitute in 1st component $\Rightarrow$ *generalized Hain–Lüst equation:*

$$\frac{d}{dr}\left[\frac{N}{rD}\frac{d\chi}{dr}\right] + \frac{1}{r}\Bigg[\rho\omega^2 - F^2 - r\left(\frac{B_\theta^2}{r^2}\right)' - \frac{4k^2B_\theta^2}{r^2 D}(B^2\rho\omega^2 - \gamma p F^2) + r\left\{\frac{2kB_\theta G}{r^2 D}((\gamma p + B^2)\rho\omega^2 - \gamma p F^2)\right\}'\Bigg]\chi = 0\,, \qquad (47)$$

with singular factors

$$N = N(r;\omega^2) \;\equiv\; \rho^2(\gamma p + B^2)(\omega^2 - \omega_A^2)\,(\omega^2 - \omega_S^2)\,,$$

$$D = D(r;\omega^2) \;\equiv\; \rho^2\omega^4 - (m^2/r^2 + k^2)(\gamma p + B^2)(\omega^2 - \omega_S^2)\,. \qquad (48)$$

- BCs:

$$\chi(0) = \chi(a) = 0 \qquad (\text{including } m = 1 \text{ at } r = 0)\,. \qquad (49)$$

MHD wave equation in cylinder with flow

- Generalizing the MHD wave equation to equilibria with background flow, exploiting the MHD spectral equation $\mathbf{G}(\boldsymbol{\xi}) + 2\mathrm{i}\rho\omega\mathbf{v}\cdot\nabla\boldsymbol{\xi} + \rho\omega^2\boldsymbol{\xi} = 0$ ($+$ a lot of algebra!)
  $\Rightarrow$ **Quadratic** *vector eigenvalue problem:*

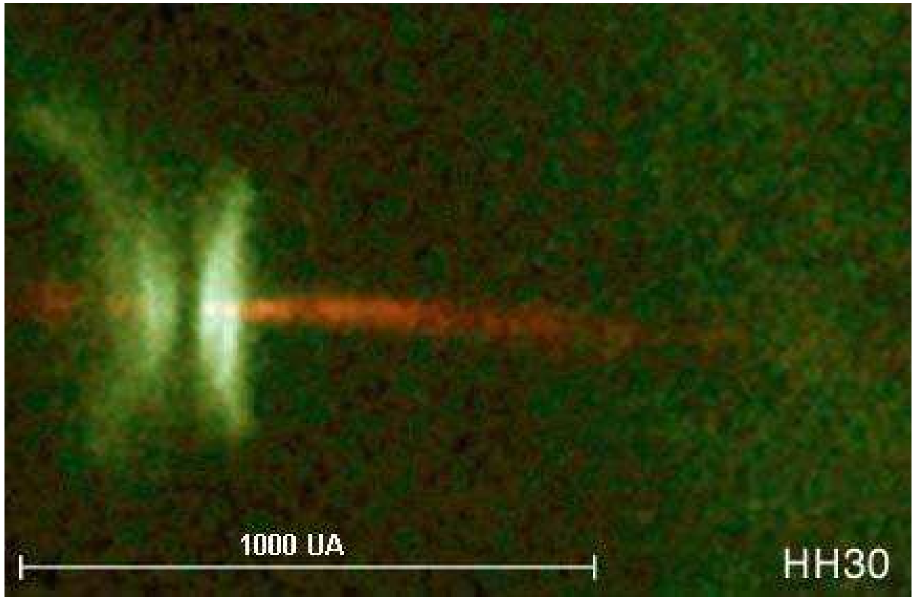
$$\left[\mathbf{F}_0 + \begin{pmatrix} -r\rho\left(\dfrac{\Phi'_{\rm gr}}{r}\right)' & -\Lambda\dfrac{G}{B} & -\Lambda\dfrac{F}{B} \\ -\Lambda\dfrac{G}{B} & 0 & 0 \\ -\Lambda\dfrac{F}{B} & 0 & 0 \end{pmatrix} - 2\rho\frac{v_\theta}{r}\widetilde{\omega} \begin{pmatrix} 0 & \dfrac{B_z}{B} & \dfrac{B_\theta}{B} \\ \dfrac{B_\theta}{B_z} & 0 & 0 \\ \dfrac{B_\theta}{B_\theta} & 0 & 0 \end{pmatrix} + \rho\widetilde{\omega}^2\,\mathbf{I} \right] \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = 0\,, \qquad (50)$$

  where $\mathbf{F}_0$ is the matrix on the LHS of the static wave equation (45), the function $\Lambda(r) \equiv \rho(v_\theta^2/r - \Phi'_{\rm gr})$ represents the deviation from static equilibrium due to rotation and gravity (or the deviation of HD equilibria from Keplerian flow), the Doppler shifted frequency $\widetilde{\omega}(r) \equiv \omega - \Omega_0(r)$, where $\Omega_0 \equiv m v_\theta/r + k v_z$, and $\mathbf{I}$ is the unit matrix.

- Elimination of η and ζ and substitution in the first component, yields again a 2nd order ODE like the Hain–Lüst equation: see Eq. (51) below.

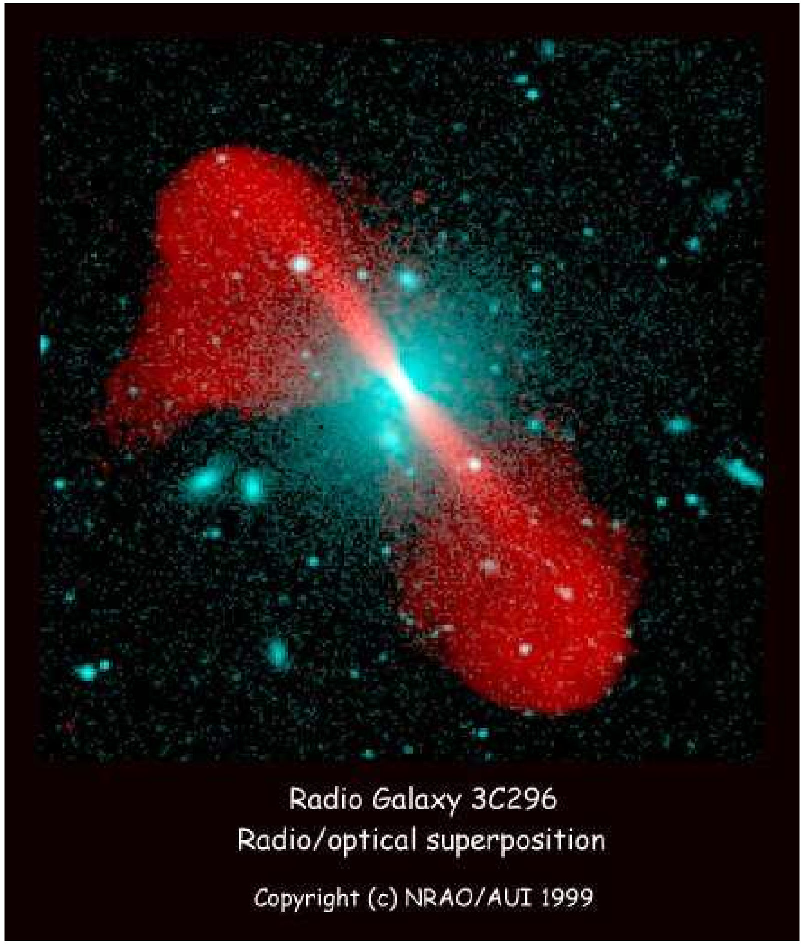
Observations

Young Stellar Object ($M_* \sim 1 M_\odot$)



disk and **jets**

Active Galactic Nucleus ($\sim 10^8 M_\odot$)



disk (optical) and **jets** (radio)

Magneto-rotational instability

- Example of **cylindrical flow.** Original references:
  – Velikhov, Soviet Phys.–JETP Lett. **36**, 995 (1959);
  – Chandrasekhar, Proc. Nat. Acad. Sci. USA **46**, 253 (1960).

- Applied to *accretion disks* by Balbus and Hawley, Astrophys. J. **376**, 214 (1991).

  Problem: how can accretion on Young Stellar Object (mass $M_* \sim M_\odot$) or Active Galactic nucleus (mass $M_* \sim 10^9 M_\odot$) occur at all on a reasonable time scale?

  – Without dissipation impossible, because disk would conserve angular momentum; some form of viscosity needed to transfer angular momentum to larger distances.

  – However, ordinary molecular viscosity much too small to produce sizeable transfer, and for turbulent increase (small-scale instabilities) no HD candidates were found.

  – It is generally assumed that the resolution of this problem involves MHD instability: **the magneto-rotational instability (MRI)**.

- Simplify the axi-symmetric (2D) representation of the disk (see sheet 4-9) even further by *neglecting vertical variations* so that a **cylindrical (1D) slice** is obtained.

  [One should object: but that is no disk at all anymore! Yet, this is how plasma-astrophysicists grapple with the problem of anomalous (turbulent) transport.]

MRI: cylindrical representation

- Generalization of Hain–Lüst equation, Book, Eq. (9.31), to **cylindrical flow with normal modes**

$$\boldsymbol{\xi} \sim \exp\left[\mathrm{i}(m\theta + kz - \omega t)\right],$$

again yields second order ODE for radial component of the plasma displacement:

$$\frac{d}{dr}\left[\frac{N}{rD}\frac{d\chi}{dr}\right] + \left[U + \frac{V}{D} + \left(\frac{W}{D}\right)'\right]\chi = 0\,, \qquad \chi \equiv r\xi\,. \tag{51}$$

[Bondeson, Iacono and Bhattacharjee, Phys. Fluids **30**, 2167 (1987);
extended with gravity: Keppens, Casse, Goedbloed, Astrophys. J. **579**, L121 (2002)]

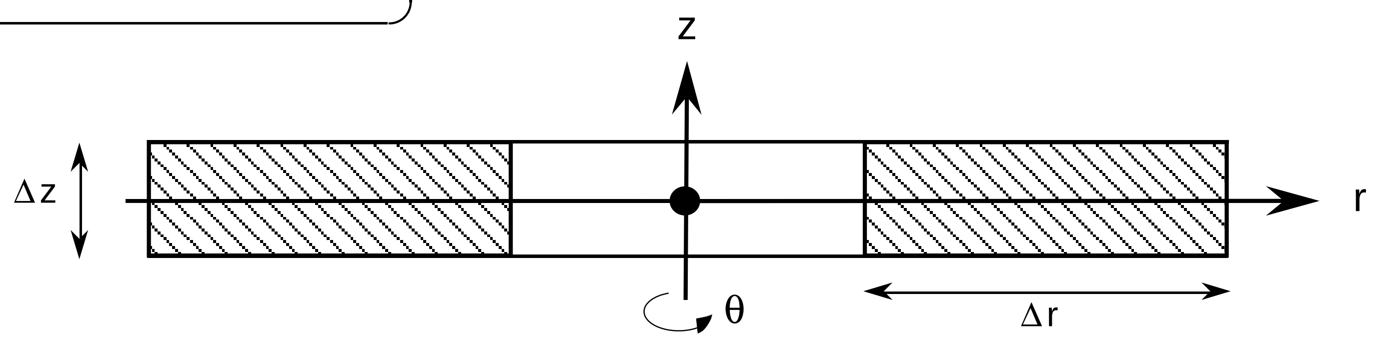
- Assumption of *small magnetic field,*

$$\beta \equiv 2p/B^2 \gg 1\,, \tag{52}$$

justifies use of this spectral equation in the *incompressible limit:*

$$\frac{d}{dr}\left[\frac{\rho\widetilde{\omega}^2 - F^2}{m^2 + k^2r^2}\, r\, \frac{d\chi}{dr}\right] - \frac{1}{r}\left[\rho\widetilde{\omega}^2 - F^2 + r\left(\frac{B_\theta^2 - \rho v_\theta^2}{r^2}\right)' + \rho'\Phi_{\rm gr}' \right.$$
$$\left. - \frac{4k^2(B_\theta F + \rho v_\theta \widetilde{\omega})^2}{(m^2 + k^2r^2)(\rho\widetilde{\omega}^2 - F^2)} - r\left(\frac{2m(B_\theta F + \rho v_\theta\widetilde{\omega})}{r(m^2 + k^2r^2)}\right)'\right]\chi = 0\,. \tag{53}$$

MRI: approximations



- Gravitational potential of compact object is approximated for cylindrical slice,

$$\Phi_{\rm gr} = -GM_*/\sqrt{r^2 + z^2} \approx -GM_*/r \,, \tag{54}$$

with *short wavelengths fitting the disk in the vertical direction:*

$$k\,\Delta z \gg 1\,. \tag{55}$$

- Incompressibility is consistent with *constant density* so that the only gravitational term, $\rho'\Phi'_{\rm gr}/r$, disappears from the spectral equation. However, $\Phi_{\rm gr}$ does not disappear from the equilibrium equation that ρ, p, B_θ, B_z, and v_θ have to satisfy,

$$(p + \tfrac{1}{2}B^2)' + \frac{1}{r}B_\theta^2 = \frac{1}{r}\rho v_\theta^2 - \rho\Phi'_{\rm gr}\,,$$

so that *stability will still be influenced by gravity.*

MRI: further approximations

- Assume *purely vertical and constant magnetic field* and *purely azimuthal velocity,*

$$B_\theta = 0\,, \quad v_z = 0 \quad \Rightarrow \quad \omega_A = kB_z/\sqrt{\rho} = \text{const}\,, \quad \Omega_0 = mv_\theta/r\,, \tag{56}$$

and restrict analysis to *vertical wave numbers k* only,

$$m = 0 \quad \Rightarrow \quad \Omega_0 = 0 \quad \Rightarrow \quad \widetilde{\omega} = \omega\,. \tag{57}$$

In the spectral equation, only ω^2 appears: $\omega = 0$ **remains the marginal point!**

$$(\omega^2 - \omega_A^2)\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right) - \frac{k^2}{r}\left[\omega^2 - \omega_A^2 - r\left(\frac{v_\theta^2}{r^2}\right)' - \frac{4\omega^2 v_\theta^2/r^2}{\omega^2 - \omega_A^2}\right]\chi = 0\,. \tag{58}$$

- Introduce *angular frequency $\Omega \equiv v_\theta/r$* and *epicyclic frequency κ,*

$$\kappa^2 \equiv \frac{1}{r^3}(r^4\Omega^2)' = 2r\Omega\Omega' + 4\Omega^2 \tag{59}$$

(Specific angular momentum $L \equiv \rho r v_\theta \equiv \rho r^2\Omega$; hence $\kappa^2 = 0 \;\Rightarrow\; L' = 0$.)

The spectral equation then becomes:

$$(\omega^2 - \omega_A^2)\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right) - \frac{k^2}{r}\left[\omega^2 - \omega_A^2 - \kappa^2(r) - \frac{4\omega_A^2\Omega^2(r)}{\omega^2 - \omega_A^2}\right]\chi = 0\,. \tag{60}$$

MRI: criteria

- Recall construction of quadratic form (sheet 7-24e):

$$(P\chi')' - Q\chi = 0 \qquad \Rightarrow \quad \int (P\chi'^2 + Q\chi^2)\, r\,dr = 0\,. \tag{61}$$

$\Rightarrow$ For eigenfunctions (oscillatory χ), we should have $Q/P < 0$ for some r.

- From Eq. (60), this gives the following *criteria for instability ($\omega^2 < 0$):*

(a) MHD ($\omega_A^2 \neq 0$):	$\omega_A^2 + \kappa^2 - 4\Omega^2 < 0$	(for some range of r) . (62)
(b) HD ($\omega_A^2 \equiv 0$):	$\kappa^2 < 0$	

- For *Keplerian rotation* (neglecting p and B on equilibrium motion):

$$\frac{1}{r}\rho v_\theta^2 = \rho \Phi'_{\rm gr} = \rho \frac{GM_*}{r^2} \quad \Rightarrow \quad \Omega^2 = \frac{GM_*}{r^3} \quad \Rightarrow \quad \kappa^2 = \frac{GM_*}{r^3} > 0\,. \tag{63}$$

$\Rightarrow$ In HD limit, opposite of (62)(b) holds, *Rayleigh's circulation criterion is satisfied: the fluid is stable* to axi-symmetric modes ($m = 0$) if $\kappa^2 \geq 0$ everywhere.

This explains the interest in MHD instabilities as candidates for turbulent increase of the dissipation processes in accretion disks.

MRI: MHD versus HD

- **MHD instability criterion in the limit $\omega_A^2 \rightarrow 0$** (magnetic field sufficiently small):

$$\kappa^2 - 4\Omega^2 \equiv 2r\Omega\Omega' < 0\,. \tag{64}$$

  This is **satisfied for Keplerian disks:** MRI works for astrophysically relevant cases! Stabilizing field contribution ($\omega_A^2 > 0$) should be small enough to maintain this effect.

- Discrepancy of HD and MHD stability results is due to *interchange of limits:*

  HD disk: $\omega_A^2 = 0\,,\ \omega^2 \rightarrow 0\,,$ MHD disk: $\omega^2 = 0\,,\ \omega_A^2 \rightarrow 0$.

  This is resolved when the *growth rates* of the instabilities are considered.

- Instead of numerically solving ODE (60), just consider radially localized modes, $\chi \sim \exp(\mathrm{i}qr),\ q\Delta r \gg 1,$ producing a *local dispersion equation:*

$$(k^2 + q^2)(\omega^2 - \omega_A^2)^2 - k^2\kappa^2(\omega^2 - \omega_A^2) - 4k^2\omega_A^2\Omega^2 = 0\,. \tag{65}$$

  Solutions for $q^2 \ll k^2$:

$$\omega^2 = \omega_A^2 + \tfrac{1}{2}\kappa^2 \pm \tfrac{1}{2}\sqrt{\kappa^4 + 16\omega_A^2\Omega^2} \approx \begin{cases} \kappa^2 + \omega_A^2(1 + 4\Omega^2/\kappa^2) \\ \omega_A^2(1 - 4\Omega^2/\kappa^2) \end{cases}, \tag{66}$$

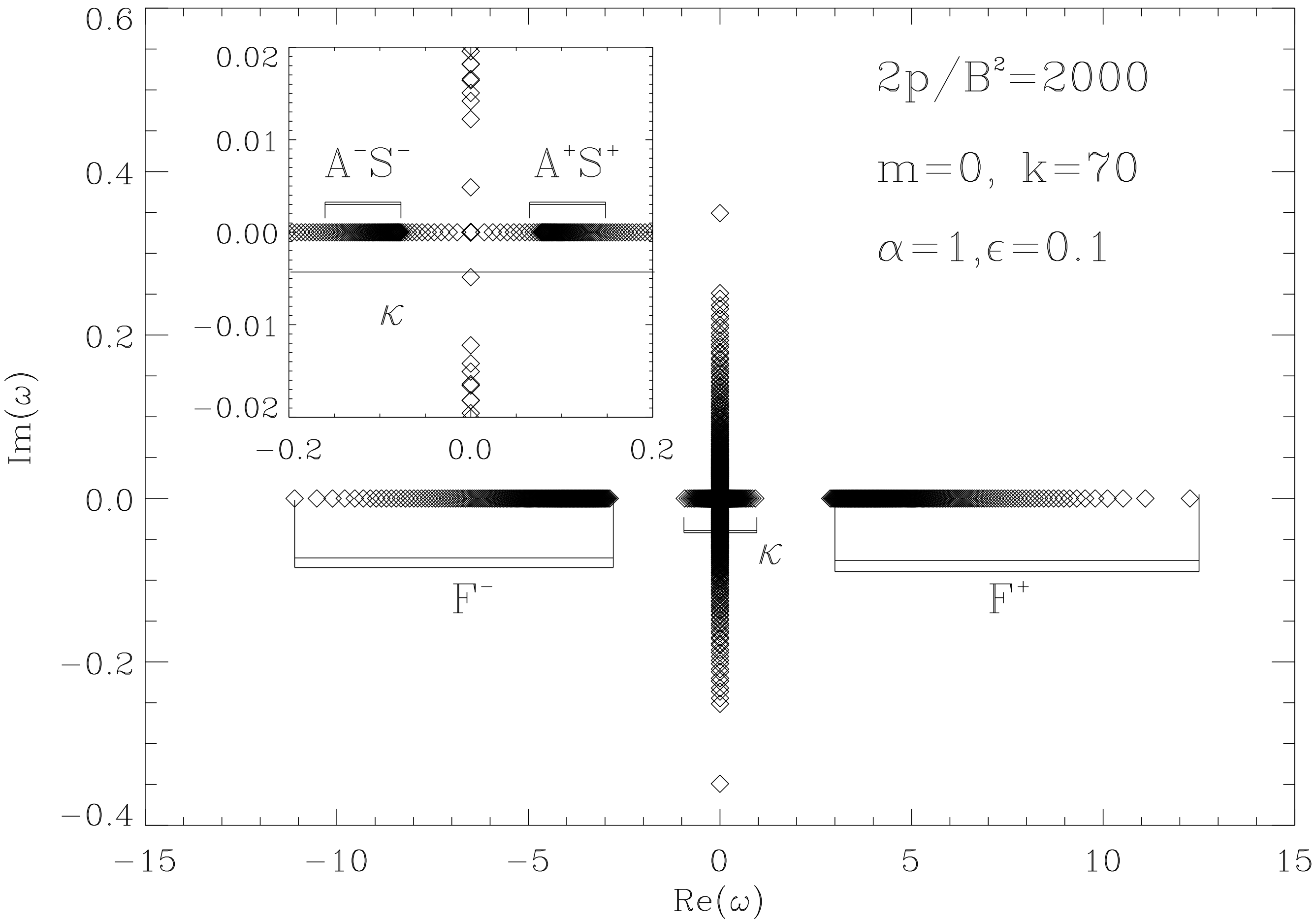
  Limit $\omega_A^2 \rightarrow 0$: (1) Rayleigh mode (HD), $\omega_+^2 \rightarrow \kappa^2 > 0$, (2) MRI (MHD), $\omega_-^2 \rightarrow 0$.

MRI: Numerical results

[Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]

- **Full spectrum with toroidal field ($m = 0$)**

  $\Rightarrow$ Discrete MRIs form cluster spectrum towards slow continuum:



MRI: Numerical results

[Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]

- **Full spectrum with toroidal field ($m = 10$)**

  $\Rightarrow$ Many more unstable sequences in the complex ω-plane: