Chapter 6: Spectral Theory

Overview

- **Intuitive approach to stability:** two viewpoints for study of stability, linearization and Lagrangian reduction;  
  [book: Sec. 6.1]

- **Force operator formalism:** equation of motion, Hilbert space, self-adjointness of the force operator;  
  [book: Sec. 6.2]

- **Quadratic forms and variational principles:** expressions for the potential energy, different variational principles, the energy principle;  
  [book: Sec. 6.4]

- **Further spectral issues:** returning to the two viewpoints;  
  [book: Sec. 6.5]

- **Extension to interface plasmas:** boundary conditions, extended variational principles, Rayleigh–Taylor instability.  
  [book: Sec. 6.6]
Two viewpoints

- How does one know whether a dynamical system is stable or not?

- Method: split the non-linear problem in *static equilibrium* (no flow) and small (linear) *time-dependent perturbations*.

- Two approaches: exploiting variational principles involving *quadratic forms* (energy), or solving *the partial differential equations themselves* (forces).
Aside: nonlinear stability

- Distinct from linear stability, involves \textit{finite amplitude displacements}:
  
  (a) system can be linearly stable, nonlinearly unstable;
  
  (b) system can be linearly unstable, nonlinearly stable (e.g. evolving towards the equilibrium states 1 or 2).

- Quite relevant for topic of magnetic confinement, but too complicated at this stage.
Linearization

• Start from ideal MHD equations:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} - \rho \nabla \Phi, \quad \mathbf{j} = \nabla \times \mathbf{B},
\]

\[
\frac{\partial p}{\partial t} = -\mathbf{v} \cdot \nabla p - \gamma p \nabla \cdot \mathbf{v},
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0,
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}).
\]

assuming model I (plasma–wall) BCs:

\[
\mathbf{n} \cdot \mathbf{v} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0 \quad (at \ the \ wall).
\]

• Linearize about static equilibrium with time-independent \( \rho_0, p_0, \mathbf{B}_0, \) and \( \mathbf{v}_0 = 0 \):

\[
\mathbf{j}_0 \times \mathbf{B}_0 = \nabla p_0 + \rho_0 \nabla \Phi, \quad \mathbf{j}_0 = \nabla \times \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0,
\]

\[
\mathbf{n} \cdot \mathbf{B}_0 = 0 \quad (at \ the \ wall).
\]
Time dependence enters through *linear perturbations* of the equilibrium:

\[\begin{align*}
\mathbf{v}(\mathbf{r}, t) &= \mathbf{v}_1(\mathbf{r}, t), \\
p(\mathbf{r}, t) &= p_0(\mathbf{r}) + p_1(\mathbf{r}, t), \\
\mathbf{B}(\mathbf{r}, t) &= \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), \\
\rho(\mathbf{r}, t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t),
\end{align*}\]  

\(\text{all, except } \mathbf{v}_1: \quad |f_1(\mathbf{r}, t)| \ll |f_0(\mathbf{r})| \)  

(8)

Inserting in Eqs. (1)–(4) yields *linear equations for* \(v_1, p_1, B_1, \rho_1\) (note strange order!):

\[\begin{align*}
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 - \rho_1 \nabla \Phi, \\
\frac{\partial p_1}{\partial t} &= -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1, \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \\
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot (\rho_0 \mathbf{v}_1).
\end{align*}\]  

(9)  

(10)  

(11)  

(12)

Since wall fixed, so is \(n\), hence BCs (5) already linear:

\[\begin{align*}
n \cdot \mathbf{v}_1 &= 0, \\
n \cdot \mathbf{B}_1 &= 0 \quad \text{(at the wall)}.
\end{align*}\]  

(13)
Lagrangian reduction

- Introduce *Lagrangian displacement vector field* \( \xi(r, t) \):
  
  plasma element is moved over \( \xi(r, t) \) away from the equilibrium position.

\[
\Rightarrow \text{Velocity is time variation of } \xi(r, t) \text{ in the comoving frame,}
\]

\[
v = \D{\xi}{D{t}} \equiv \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi,
\]

involving the Lagrangian time derivative \( \D{D}{t} \) (co-moving with the plasma).
• Linear (first order) part relation yields

\[ \mathbf{v} \approx \mathbf{v}_1 = \frac{\partial \xi}{\partial t}, \]  

(15)

only involving the Eulerian time derivative (fixed in space).

• Inserting in linearized equations, can directly integrate (12):

\[ \frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1) \quad \Rightarrow \quad \rho_1 = -\nabla \cdot (\rho_0 \xi). \]  

(16)

Similarly linearized energy (10) and induction equation (11) integrate to

\[ p_1 = -\xi \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \xi, \]  

(17)

\[ \mathbf{B}_1 = \nabla \times (\xi \times \mathbf{B}_0) \quad \text{(automatically satisfies } \nabla \cdot \mathbf{B}_1 = 0). \]  

(18)

• Inserting these expressions into linearized momentum equation yields

\[ \rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}(p_1(\xi), \mathbf{B}_1(\xi), \rho_1(\xi)). \]  

(19)

\[ \Rightarrow \quad \text{Equation of motion with force operator } \mathbf{F}. \]
Force Operator formalism

• Insert explicit expression for \( \mathbf{F} \) ⇒ *Newton’s law for plasma element:*

\[
\mathbf{F}(\xi) \equiv -\nabla \pi - \mathbf{B} \times (\nabla \times \mathbf{Q}) + (\nabla \times \mathbf{B}) \times \mathbf{Q} + (\nabla \Phi) \nabla \cdot (\rho \mathbf{\xi}) = \rho \frac{\partial^2 \mathbf{\xi}}{\partial t^2}, \tag{20}
\]

with change of notation (so that we can drop subscripts \(_0\) and \(_1\)):

\[
\pi \equiv p_1 = -\gamma p \nabla \cdot \mathbf{\xi} - \mathbf{\xi} \cdot \nabla p, \tag{21}
\]

\[
\mathbf{Q} \equiv \mathbf{B}_1 = \nabla \times (\mathbf{\xi} \times \mathbf{B}). \tag{22}
\]

• Geometry (plane slab, cylinder, torus, etc.) defined by shape wall, through BC:

\[
\mathbf{n} \cdot \mathbf{\xi} = 0 \quad \text{(at the wall).} \tag{23}
\]

• Now count: three 2nd order PDEs for vector \( \mathbf{\xi} \) ⇒ sixth order *Lagrangian* system; originally: eight 1st order PDEs for \( \rho_1, \mathbf{v}_1, p_1, \mathbf{B}_1 \) ⇒ eight order *Eulerian* system.

• Third component of \( \mathbf{B}_1 \) is redundant (\( \nabla \cdot \mathbf{B}_1 = 0 \)), and equation for \( \rho_1 \) produces trivial Eulerian entropy mode \( \omega_E = 0 \) (with \( \rho_1 \neq 0 \), but \( \mathbf{v}_1 = 0, p_1 = 0, \mathbf{B}_1 = 0 \)).

⇒ Neglecting this mode, Lagrangian and Eulerian representation equivalent.
Ideal MHD spectrum

- Consider normal modes:

\[ \xi(r, t) = \hat{\xi}(r) e^{-i\omega t}. \]  \hspace{1cm} (24)

\[ \Rightarrow \text{Equation of motion becomes eigenvalue problem:} \]

\[ F(\hat{\xi}) = -\rho \omega^2 \hat{\xi}. \]  \hspace{1cm} (25)

- For given equilibrium, collection of eigenvalues \( \{\omega^2\} \) is spectrum of ideal MHD.

\[ \Rightarrow \text{Generally both discrete and continuous ('improper') eigenvalues.} \]

- The operator \( \rho^{-1}F \) is self-adjoint (for fixed boundary).

\[ \Rightarrow \text{The eigenvalues } \omega^2 \text{ are real.} \]

\[ \Rightarrow \text{Same mathematical structure as for quantum mechanics!} \]
• Since $\omega^2$ real, $\omega$ itself either real or purely imaginary

$\Rightarrow$ In ideal MHD, only stable waves ($\omega^2 > 0$) or exponential instabilities ($\omega^2 < 0$):

(a) \hspace{1cm} \hspace{1cm} (b)

$\Rightarrow$ Crudely, $F(\hat{\xi}) \sim -\hat{\xi}$ for $\omega^2 > 0$ and $\sim \hat{\xi}$ for $\omega^2 < 0$ (cf. intuitive picture).
Dissipative MHD

- In resistive MHD, operators no longer self-adjoint, eigenvalues $\omega^2$ complex.

  $\Rightarrow$ **Stable, damped waves and ‘overstable’ modes ($\equiv$ instabilities):**

\[
\begin{align*}
\omega &= -\sigma - iv \\
&\quad \sigma - iv \\
\omega &= -\sigma + iv \\
&\quad \sigma + iv
\end{align*}
\]
Stability in ideal MHD

- For ideal MHD, transition from stable to unstable through $\omega^2 = 0$: *marginal stability.

  ⇒ Study marginal equation of motion

  $\mathbf{F}(\hat{\xi}) = 0$. \hfill (26)

  ⇒ In general, this equation has no solution since $\omega^2 = 0$ is not an eigenvalue.

- Can vary equilibrium parameters until zero eigenvalue is reached, e.g. in tokamak stability analysis, the parameters $\beta \equiv 2\mu_0 p/B^2$ and ‘safety factor’ $q_1 \sim 1/I_p$.

  ⇒ Find critical curve along which $\omega^2 = 0$ is an eigenvalue:

  ⇒ this curve separates stable from unstable parameter states.
Physical meaning of the terms of $F$

- Rearrange terms:
  \[
  F(\xi) = \nabla (\gamma p \nabla \cdot \xi) - B \times (\nabla \times Q) + \nabla (\xi \cdot \nabla p) + j \times Q + \nabla \Phi \nabla \cdot (\rho \xi).
  \]  
  (27)

  First two terms (with $\gamma p$ and $B$) present in **homogeneous equilibria**, last three terms only in **inhomogeneous equilibria** (when $\nabla p$, $j$, $\nabla \Phi \neq 0$).

- Homogeneous equilibria
  \[\Rightarrow\] isotropic force $\nabla (\gamma p \nabla \cdot \xi)$: compressible sound waves;
  \[\Rightarrow\] anisotropic force $B \times (\nabla \times Q)$: field line bending Alfvén waves;
  \[\Rightarrow\] waves always stable (see below).

- Inhomogeneous equilibria have **pressure gradients, currents, gravity**
  \[\Rightarrow\] potential sources for instability: will require extensive study!
Homogeneous case

- Sound speed $c \equiv \sqrt{\gamma p/\rho}$ and Alfvén speed $b \equiv B/\sqrt{\rho}$ constant, so that
  \[ \rho^{-1}F(\hat{\xi}) = c^2 \nabla \nabla \cdot \hat{\xi} + b \times (\nabla \times (\nabla \times (b \times \hat{\xi}))) = -\omega^2 \hat{\xi}. \]  
  \[ (28) \]

  Plane wave solutions $\hat{\xi} \sim \exp(ik \cdot r)$ give
  \[ \rho^{-1}F(\hat{\xi}) = \left[ -(k \cdot b)^2 I - (b^2 + c^2) kk + k \cdot b (kb + bk) \right] \cdot \hat{\xi} = -\omega^2 \hat{\xi} \]  
  \[ (29) \]

  $\Rightarrow$ recover the stable waves of Chapter 5.

- Recall: slow, Alfvén, fast eigenvectors $\hat{\xi}_s, \hat{\xi}_A, \hat{\xi}_f$ form orthogonal triad
  $\Rightarrow$ can decompose any vector in combination of these 3 eigenvectors of $F$;
  $\Rightarrow$ eigenvectors span whole space: Hilbert space of plasma displacements.

- Extract Alfvén wave (transverse incompressible $k \cdot \xi = 0$, $B$ and $k$ along $z$):
  \[ \rho^{-1}F_y = b^2 \frac{\partial^2 \hat{\xi}_y}{\partial z^2} = -k_z^2 b^2 \hat{\xi}_y = \frac{\partial^2 \hat{\xi}_y}{\partial t^2} = -\omega^2 \hat{\xi}_y, \]  
  \[ (30) \]

  $\Rightarrow$ Alfvén waves, $\omega^2 = \omega_A^2 \equiv k_z^2 b^2$, dynamical centerpiece of MHD spectral theory.
Hilbert space

- Consider plasma volume \( V \) enclosed by wall \( W \), with two displacement vector fields (satisfying the BCs):

\[
\xi = \xi(r, t) \quad \text{(on } V\text{)}, \quad \text{where } n \cdot \xi = 0 \quad \text{(at } W\text{)},
\]
\[
\eta = \eta(r, t) \quad \text{(on } V\text{)}, \quad \text{where } n \cdot \eta = 0 \quad \text{(at } W\text{)}. \tag{31}
\]

Define inner product (weighted by the density):

\[
\langle \xi, \eta \rangle \equiv \frac{1}{2} \int \rho \xi^\ast \cdot \eta \, dV, \tag{32}
\]

and associated norm

\[
\| \xi \| \equiv \langle \xi, \xi \rangle^{1/2}. \tag{33}
\]

- All functions with finite norm \( \| \xi \| < \infty \) form linear function space, a Hilbert space.

\( \Rightarrow \) Force operator \( \mathbf{F} \) is linear operator in Hilbert space of vector displacements.
Analogy with quantum mechanics

• Recall Schrödinger equation for wave function $\psi$:

$$H\psi = E\psi.$$ \hfill (34)

$\Rightarrow$ Eigenvalue equation for Hamiltonian $H$ with eigenvalues $E$ (energy levels).

- $E = 0$ (continuous)
- $E > 0$ (discrete)

$\Rightarrow$ Spectrum of eigenvalues $\{E\}$ consists of discrete spectrum for bound states ($E < 0$) and continuous spectrum for free particle states ($E > 0$).

$\Rightarrow$ Norm $\|\psi\| \equiv \langle \psi, \psi \rangle^{1/2}$ gives probability to find particle in the volume.

• Central property in quantum mechanics: Hamiltonian $H$ is \textit{self-adjoint} linear operator in Hilbert space of wave functions,

$$\langle \psi_1, H\psi_2 \rangle = \langle H\psi_1, \psi_2 \rangle.$$ \hfill (35)
How about the force operator $F$? Is it self-adjoint and, if so, what does it mean?

Self-adjointness is related to energy conservation. For example, finite norm of $\xi$, or its time derivative $\dot{\xi}$, means that the kinetic energy is bounded:

$$K \equiv \frac{1}{2} \int \rho \mathbf{v}^2 \, dV \approx \frac{1}{2} \int \rho \dot{\mathbf{x}}^2 \, dV = \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle \equiv \| \dot{\mathbf{x}} \|_2^2.$$  \hspace{1cm} (36)

Consequently, the potential energy (related to $F$, as we will see) is also bounded.

The good news: *force operator* $\rho^{-1}F$ is self-adjoint linear operator in Hilbert space of plasma displacement vectors:

$$\langle \eta, \rho^{-1}F(\xi) \rangle \equiv \frac{1}{2} \int \eta^* \cdot F(\xi) \, dV = \frac{1}{2} \int \xi \cdot F(\eta^*) \, dV \equiv \langle \rho^{-1}F(\eta), \xi \rangle.$$  \hspace{1cm} (37)

$\Rightarrow$ The mathematical analogy with quantum mechanics is complete.

And the bad news: the proof of that central property is horrible!
Proving self-adjointness

- Proving

\[ \int \eta^* \cdot F(\xi) \, dV = \int \xi \cdot F(\eta^*) \, dV \]

involves lots of tedious vector manipulations, with two returning ingredients:

- use of equilibrium relations \( j \times B = \nabla p + \rho \nabla \Phi \), \( j = \nabla \times B \), \( \nabla \cdot B = 0 \);
- manipulation of volume integral to symmetric part in \( \eta \) and \( \xi \) and divergence term, which transforms into surface integral on which BCs are applied.

- Notational conveniences:
  - defining magnetic field perturbations associated with \( \xi \) and \( \eta \),
    \[ Q(r) \equiv \nabla \times (\xi \times B) \quad \text{(on } V) \],
  \[ R(r) \equiv \nabla \times (\eta \times B) \quad \text{(on } V) \];
  \[ \eta^* \cdot F(\xi) + \text{complex conjugate} \Rightarrow \eta \cdot F(\xi). \]
• Omitting intermediate steps [see book: Sec. 6.2.3], we get useful, near-final result:

\[
\int \eta \cdot F(\xi) \, dV = - \int \left\{ \gamma p \nabla \cdot \xi \nabla \cdot \eta + Q \cdot R + \frac{1}{2} \nabla p \cdot (\xi \nabla \cdot \eta + \eta \nabla \cdot \xi) \\
+ \frac{1}{2} j \cdot (\eta \times Q + \xi \times R) - \frac{1}{2} \nabla \Phi \cdot [\eta \nabla \cdot (\rho \xi) + \xi \nabla \cdot (\rho \eta)] \right\} \, dV \\
+ \int n \cdot \eta \left[ \gamma p \nabla \cdot \xi + \xi \cdot \nabla p - B \cdot Q \right] \, dS. 
\]

This symmetric expression is general, valid for all model I–V problems.

• Restricting to model I (wall on the plasma), surface integrals vanish because of BC \( n \cdot \xi = 0 \), and self-adjointness results:

\[
\int \left\{ \eta \cdot F(\xi) - \xi \cdot F(\eta) \right\} \, dV = \int \left\{ n \cdot \eta \left[ \gamma p \nabla \cdot \xi + \xi \cdot \nabla p - B \cdot Q \right] \\
- n \cdot \xi \left[ \gamma p \nabla \cdot \eta + \eta \cdot \nabla p - B \cdot R \right] \right\} \, dS = 0, \quad \text{QED.} \quad (40)
\]

• Proof of self-adjointness for model II, etc. is rather straightforward now. It involves manipulating the surface term, using the pertinent BCs, to volume integral over the external vacuum region + again a vanishing surface integral over the wall.
Important result

• The eigenvalues of $\rho^{-1} F$ are real.

• Proof
  – Consider pair of eigenfunction $\xi_n$ and eigenvalue $-\omega_n^2$:
    $$\rho^{-1} F(\xi_n) = -\omega_n^2 \xi_n;$$
  – take complex conjugate:
    $$\rho^{-1} F^*(\xi_n^*) = \rho^{-1} F(\xi_n^*) = -\omega_n^{2*} \xi_n^*;$$
  – multiply 1st equation with $\xi_n^*$ and 2nd with $\xi_n$, subtract, integrate over volume, and exploit self-adjointness:
    $$0 = (\omega_n^2 - \omega_n^{2*}) \| \xi \|^2 \quad \Rightarrow \quad \omega_n^2 = \omega_n^{2*}, \quad QED.$$ 

• Consequently, $\omega^2$ either $\geq 0$ (stable) or $< 0$ (unstable): everything falls in place!
Quadratic forms for potential energy

- Alternative representation is obtained from expressions for kinetic energy $K$ and potential energy $W$, exploiting energy conservation: $H \equiv W + K = \text{const}$. 

- (a) Use expression for $K$ (already encountered) and equation of motion:

$$\frac{dK}{dt} \equiv \frac{d}{dt} \left[ \frac{1}{2} \int \rho |\dot{\xi}|^2 \, dV \right] = \int \rho \dot{\xi}^* \cdot \ddot{\xi} \, dV = \int \dot{\xi}^* \cdot F(\xi) \, dV. \quad (41)$$

(b) Exploit energy conservation and self-adjointness:

$$\frac{dW}{dt} = -\frac{dK}{dt} = -\frac{1}{2} \int \left[ \dot{\xi}^* \cdot F(\xi) + \xi^* \cdot F(\xi) \right] \, dV = \frac{d}{dt} \left[ -\frac{1}{2} \int \xi^* \cdot F(\xi) \, dV \right].$$

(c) Integration yields linearized potential energy expression:

$$W = -\frac{1}{2} \int \xi^* \cdot F(\xi) \, dV. \quad (42)$$

- Intuitive meaning of $W$: potential energy increase from work done against force $F$ (hence, minus sign), with $\frac{1}{2}$ since displacement builds up from 0 to final value.
• **More useful form of** \( W \) **follows from earlier expression** \([39]\) (with \( \eta \to \xi^* \)) **used in self-adjointness proof:**

\[
W = \frac{1}{2} \int \left[ \gamma p |\nabla \cdot \xi|^2 + |Q|^2 + (\xi \cdot \nabla p) \nabla \cdot \xi^* + j \cdot \xi^* \times Q \\
- (\xi^* \cdot \nabla \Phi) \nabla \cdot (\rho \xi) \right] dV ,
\]  
\( (43) \)

(to be used with model I BC)

\[
\mathbf{n} \cdot \xi = 0 \quad \text{(at the wall)} .
\]  
\( (44) \)

• Earlier discussion on stability can now be completed:
  
  – first two terms (acoustic and magnetic energy) positive definite  
    \( \Rightarrow \) homogeneous plasma stable;
  
  – last three terms (pressure gradient, current, gravity) can have either sign  
    \( \Rightarrow \) **inhomogeneous plasma may be unstable** (requires extensive analysis).
Three variational principles

• Recall three levels of description with *differential equations*:
  (a) Equation of motion (20): \( F(\xi) = \rho \ddot{\xi} \Rightarrow \text{full dynamics}; \)
  (b) Normal mode equation (25): \( F(\hat{\xi}) = -\rho \omega^2 \hat{\xi} \Rightarrow \text{spectrum of modes}; \)
  (c) Marginal equation of motion (26): \( F(\hat{\xi}) = 0 \Rightarrow \text{stability only}. \)

• Exploiting quadratic forms \( W \) and \( K \) yields *three variational counterparts*:
  (a) *Hamilton’s principle* \( \Rightarrow \text{full dynamics}; \)
  (b) *Rayleigh–Ritz spectral principle* \( \Rightarrow \text{spectrum of modes}; \)
  (c) *Energy principle* \( \Rightarrow \text{stability only}. \)
(a) Hamilton’s principle

- Variational formulation of linear dynamics in terms of Lagrangian:

The evolution of the system from time $t_1$ to time $t_2$ through the perturbation $\xi(r, t)$ is such that the variation of the integral of the Lagrangian vanishes,

$$\delta \int_{t_1}^{t_2} L \, dt = 0, \quad L \equiv K - W,$$

with

$$K = K[\dot{\xi}] = \frac{1}{2} \int \rho \dot{\xi}^* \cdot \dot{\xi} \, dV,$$

$$W = W[\xi] = -\frac{1}{2} \int \xi^* \cdot F(\xi) \, dV.$$

- Minimization (see Goldstein on classical fields) gives Euler–Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}_j} + \sum_k \frac{d}{dx_k} \frac{\partial L}{\partial (\partial \xi_j / \partial x_k)} - \frac{\partial L}{\partial \xi_j} = 0 \quad \Rightarrow \quad F(\xi) = \rho \frac{\partial^2 \xi}{\partial t^2},$$

which is the equation of motion, QED.
Consider quadratic forms $W$ and $K$ (here $I$) for normal modes $\hat{\xi} e^{-i\omega t}$:

$$F(\hat{\xi}) = -\rho \omega^2 \hat{\xi} \quad \Rightarrow \quad -\frac{1}{2} \int \hat{\xi}^* \cdot F(\hat{\xi}) \, dV = \omega^2 \cdot \frac{1}{2} \int \rho \hat{\xi}^* \cdot \hat{\xi} \, dV \equiv W[\hat{\xi}] = I[\hat{\xi}] \cdot$$

This gives

$$\omega^2 = \frac{W[\hat{\xi}]}{I[\hat{\xi}]} \text{ for normal modes}. \quad (47)$$

True, but useless: just conclusion a posteriori on $\xi$ and $\omega^2$, no recipe to find them.

- Obtain recipe by turning this into Rayleigh–Ritz variational expression for eigenvalues:

  Eigenfunctions $\xi$ of the operator $\rho^{-1}F$ make the Rayleigh quotient

  $$\Lambda[\xi] \equiv \frac{W[\xi]}{I[\xi]} \quad (48)$$

  stationary; eigenvalues $\omega^2$ are the stationary values of $\Lambda$.

  $\Rightarrow$ Practical use: approximate eigenvalues/eigenfunctions by minimizing $\Lambda$ over linear combination of pre-chosen set of trial functions $(\eta_1, \eta_2, \ldots \eta_N)$. 

(c) Energy principle for stability

- Since \( I \equiv \| \xi \|^2 \geq 0 \), Rayleigh–Ritz variational principle offers possibility of testing for stability by *inserting trial functions in \( W \):
  - If \( W[\xi] < 0 \) for single \( \xi \), at least one eigenvalue \( \omega^2 < 0 \) and system is *unstable*;
  - If \( W[\xi] > 0 \) for all \( \xi \)s, eigenvalues \( \omega^2 < 0 \) do not exist and system is *stable*.

- \( \Rightarrow \) **Energy principle**: *An equilibrium is stable if (sufficient) and only if (necessary)*

\[
W[\xi] > 0
\]  

(49)

*for all displacements \( \xi(\mathbf{r}) \) that are bound in norm and satisfy the BCs.*

- Summarizing, the variational approach offers three methods to determine stability:
  1. Guess a trial function \( \xi(\mathbf{r}) \) such that \( W[\xi] < 0 \) for a certain system  
     \( \Rightarrow \) *necessary stability (\( \equiv \) sufficient instability) criterium*;
  2. Investigate sign of \( W \) with complete set of arbitrarily normalized trial functions  
     \( \Rightarrow \) *necessary + sufficient stability criterium*;
  3. Minimize \( W \) with complete set of properly normalized functions (i.e. with \( I[\xi] \), related to kinetic energy)  
     \( \Rightarrow \) *complete spectrum of (discrete) eigenvalues.*
### Returning to the two viewpoints

- Spectral theory elucidates analogies between different parts of physics:

  \[
  \begin{align*}
  &\text{MHD} & \iff & \text{Linear analysis} & \iff & \text{QM} \\
  &\text{Force operator} & \iff & \text{Differential equations} & \iff & \text{Schrödinger picture} \\
  &\text{Energy principle} & \iff & \text{Quadratic forms} & \iff & \text{Heisenberg picture}
  \end{align*}
  \]

  The analogy is through mathematics, not through physics!

- Linear operators in Hilbert space as such have nothing to do with quantum mechanics. Mathematical formulation by Hilbert (1912) preceded it by more than a decade. Essentially, the two ‘pictures’ are just translation to physics of the generalization of linear algebra to infinite-dimensional vector spaces (Moser, 1973).

- Whereas quantum mechanics applies to rich arsenal of spherically symmetric systems (symmetry with respect to rotation groups), in MHD the constraint \( \nabla \cdot \mathbf{B} = 0 \) forbids spherical symmetry and implies much less obvious symmetries.

  \[\Rightarrow\] Application of group theory to MHD is still in its infancy.
Two ‘pictures’ of MHD spectral theory:

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<td>(‘Heisenberg’)</td>
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<td><strong>Equation of motion:</strong></td>
<td><strong>Hamilton’s principle:</strong></td>
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<td>( F(\xi) = \rho \frac{\partial^2 \xi}{\partial t^2} )</td>
<td>( \delta \int_{t_1}^{t_2} \left( K[\dot{\xi}] - W[\xi] \right) dt = 0 ) ⇒ <strong>Full dynamics:</strong> ( \xi(r, t) )</td>
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<td>( F(\xi) = -\rho \omega^2 \xi )</td>
<td>( \delta \frac{W[\xi]}{I[\xi]} = 0 ) ⇒ <strong>Spectrum</strong> ( { \omega^2 } ) &amp; eigenf. ( { \xi(r) } )</td>
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<tr>
<td>( F(\xi) = 0 )</td>
<td>( W[\xi] &gt; 0 ) ⇒ <strong>Stability</strong> ( (\tilde{Y}_n) ) &amp; trial ( \xi(r) )</td>
</tr>
</tbody>
</table>

\( F(\xi) \) = force functional; \( K[\dot{\xi}] \) = kinetic energy; \( W[\xi] \) = potential energy; \( \delta \int \) = variation; \( \xi(r, t) \) = field function; \( \omega^2 \) = eigenfrequency; \( (\tilde{Y}_n) \) = trial function.
Why does the water fall out of the glass?

- Apply spectral theory and energy principle to simple fluid (no magnetic field) with \textit{varying density in external gravitational field}. Equilibrium: \( \nabla p = -\rho \nabla \Phi = \rho g \).

\[
W^f = \frac{1}{2} \int \left[ \gamma p |\nabla \cdot \xi|^2 + (\xi \cdot \nabla p) \nabla \cdot \xi^* - (\xi^* \cdot \nabla \Phi) \nabla \cdot (\rho \xi) \right] dV
\]

\[
= \frac{1}{2} \int \left[ \gamma p |\nabla \cdot \xi|^2 + \rho g \cdot (\xi \nabla \cdot \xi^* + \xi^* \nabla \cdot \xi) + g \cdot \xi^* (\nabla \rho) \cdot \xi \right] dV. \quad (50)
\]

Without gravity, fluid is stable since only positive definite first term remains.

- \textit{Plane slab}, \( p(x), \rho(x), g = -ge_x \Rightarrow \) equilibrium: \( p' = -\rho g \).

\[
W^f = \frac{1}{2} \int \left[ \gamma p |\nabla \cdot \xi|^2 - \rho g (\xi_x \nabla \cdot \xi^* + \xi^*_x \nabla \cdot \xi) - \rho' g |\xi_x|^2 \right] dV. \quad (51)
\]

- Energy principle according to \textit{method (1)} illustrated by exploiting incompressible trial functions, \( \nabla \cdot \xi = 0 \):

\[
W^f = -\frac{1}{2} \int \rho' g |\xi_x|^2 dV \geq 0 \quad \Rightarrow \quad \rho' g \leq 0 \quad \text{(everywhere)} \quad (52)
\]

\( \Rightarrow \) \textit{Necessary stability criterion: lighter fluid should be on top of heavier fluid.}
Much sharper stability condition from energy principle according to method (2), where all modes (also compressible ones) are considered. Rearrange terms in Eq. (51):

\[ W^f = \frac{1}{2} \int \left[ \gamma p \left| \nabla \cdot \xi - \frac{\rho g}{\gamma p} \xi_x \right|^2 - \left( \rho' g + \frac{\rho^2 g^2}{\gamma p} \right) |\xi_x|^2 \right] dV. \] (53)

Since \( \xi_y \) and \( \xi_z \) only appear in \( \nabla \cdot \xi \), minimization with respect to them is trivial:

\[ \nabla \cdot \xi = \frac{\rho g}{\gamma p} \xi_x. \] (54)

\[ \Rightarrow \text{ Necessary and sufficient stability criterion:} \]

\[ \rho' g + \frac{\rho^2 g^2}{\gamma p} \leq 0 \quad \text{(everywhere)} \] (55)

Actually, we have now derived conditions for stability with respect to internal modes. Original water-air system requires extended energy principle with two-fluid interface (model II*), permitting description of external modes: our next subject. Physics will be the same: density gradient becomes density jump, that should be negative at the interface (light fluid above) for stability.
**Interfaces**

- So far, plasmas bounded by rigid wall (model I). Most applications require interface:
  - In tokamaks, very low density close to wall (created by ‘limiter’) is effectively vacuum ⇒ *plasma–vacuum system* (model II);
  - In astrophysics, frequently density jump (e.g. to low-density force-free plasma) ⇒ *plasma–plasma system* (model II*).

- Model II: split vacuum magnetic field in equilibrium part $\hat{B}$ and perturbation $\hat{Q}$.

  **Equilibrium:** $\nabla \times \hat{B} = 0$, $\nabla \cdot \hat{B} = 0$, with BCs
  \[
  n \cdot B = n \cdot \hat{B} = 0, \quad \left[ p + \frac{1}{2}B^2 \right] = 0 \quad (at \ interface \ S),
  \]
  \[
  n \cdot \hat{B} = 0 \quad (at \ outer \ wall \ \hat{W}).
  \]

  **Perturbations:** $\nabla \times \hat{Q} = 0$, $\nabla \cdot \hat{Q} = 0$, with two non-trivial BCs connecting $\hat{Q}$ to the plasma variable $\xi$ at the interface, and one BC at the wall:
  \[
  1^{st} \ interface \ cond., \quad 2^{nd} \ interface \ cond. \quad (at \ interface \ S),
  \]
  \[
  n \cdot \hat{Q} = 0 \quad (at \ outer \ wall \ \hat{W}).
  \]

Explicit derivation of interface conditions (58) below: Eqs. (62) and (63).
Boundary conditions for interface plasmas

- Need expression for perturbation of the normal \( n \) to the interface.

- Integrating Lagrangian time derivative of line element (see Chap. 4) yields:
  \[
  \frac{D(dl)}{Dt} = dl \cdot (\nabla v) \implies dl \approx dl_0 \cdot (1 + \nabla \xi).
  \]

- For \( dl \) lying in the boundary surface:
  \[\[0 = n \cdot dl \approx dl_0 \cdot (1 + \nabla \xi) \cdot (n_0 + n_{1L}) \approx dl_0 \cdot [(\nabla \xi) \cdot n_0 + n_{1L}]\.
\]

  \[\Rightarrow \text{Lagrangian perturbation: } n_{1L} = - (\nabla \xi) \cdot n_0 + \lambda, \text{ with vector } \lambda \perp dl_0.\]

Since \( dl_0 \) has arbitrary direction in unperturbed surface, \( \lambda \) must be \( \parallel n_0 : \lambda = \mu n_0.\)

Since \( |n| = |n_0| = 1, \) we have \( n_0 \cdot n_{1L} = 0, \) so that \( \mu = n_0 \cdot (\nabla \xi) \cdot n_0.\)

This provides the Lagrangian perturbation of the normal:

\[
    n_{1L} = - (\nabla \xi) \cdot n_0 + n_0 \cdot (\nabla \xi) \cdot n_0 = n_0 \times \{n_0 \times [(\nabla \xi) \cdot n_0]\}. \quad (60)
\]
• Original BCs for model II come from jump conditions of Chap. 4:
  
  (a) \( n \cdot B = n \cdot \hat{B} = 0 \) \hspace{1em} (at plasma–vacuum interface) ,
  
  (b) \( [p + \frac{1}{2}B^2] = 0 \) \hspace{1em} (at plasma–vacuum interface) .

Also need Lagrangian perturbation of magnetic field \( B \) and pressure \( p \) at perturbed boundary position \( r \), evaluated to first order:

\[
B|_r \approx (B_0 + Q + \xi \cdot \nabla B_0)|_{r_0} ,
\]

\[
p|_r \approx (p_0 + \pi + \xi \cdot \nabla p_0)|_{r_0} = (p_0 - \gamma p_0 \nabla \cdot \xi)|_{r_0} .
\] (61)

• Insert Eqs. (60) and (61) into first part of above BC (a):

\[
0 = n \cdot B \approx [n_0 - (\nabla \xi) \cdot n_0 + n_0 n_0 \cdot (\nabla \xi) \cdot n_0] \cdot (B_0 + Q + \xi \cdot \nabla B_0)
\approx -B_0 \cdot (\nabla \xi) \cdot n_0 + n_0 \cdot Q + \xi \cdot (\nabla B_0) \cdot n_0 = -n_0 \cdot \nabla \times (\xi \times B_0) + n_0 \cdot Q .
\]

Automatically satisfied since \( Q \equiv \nabla \times (\xi \times B_0) \). However, same derivation for second part of BC (a) gives 1st interface condition relating \( \xi \) and \( \hat{Q} \):

\[
n \cdot \nabla \times (\xi \times \hat{B}) = n \cdot \hat{Q} \hspace{1em} (at \ plasma–vacuum \ interface \ S) .
\] (62)

• Inserting Eqs. (61) into BC (b) yields 2nd interface condition relating \( \xi \) and \( \hat{Q} \):

\[
-\gamma p \nabla \cdot \xi + B \cdot Q + \xi \cdot \nabla\left(\frac{1}{2}B^2\right) = \hat{B} \cdot \hat{Q} + \xi \cdot \nabla\left(\frac{1}{2}\hat{B}^2\right) \hspace{1em} (at \ S) .
\] (63)
Extended energy principle

- Proof *self-adjointness* continues from integral (*39*) for $\xi$, $\eta$, connected with vacuum ‘extensions’ $\hat{Q}$, $\hat{R}$ through BCs (*59*), (*62*), (*63*), giving symmetric quadratic form.

- Putting $\eta = \xi^*$, $\hat{R} = \hat{Q}^*$ in integrals gives potential energy for interface plasmas:

$$W[\xi, \hat{Q}] = -\frac{1}{2} \int \xi^* \cdot F(\xi) \, dV = W^p[\xi] + W^s[\xi_n] + W^v[\hat{Q}], \quad (64)$$

where

$$W^p[\xi] = \frac{1}{2} \int \left[ \gamma p |\nabla \cdot \xi|^2 + |Q|^2 + (\xi \cdot \nabla p) \nabla \cdot \xi^* + j \cdot \xi^* \times Q \right. \left. - (\xi^* \cdot \nabla \Phi) \nabla \cdot (\rho \xi) \right] \, dV, \quad (65)$$

$$W^s[\xi_n] = \frac{1}{2} \int |n \cdot \xi|^2 n \cdot [\nabla (p + \frac{1}{2} B^2)] \, dS, \quad (66)$$

$$W^v[\hat{Q}] = \frac{1}{2} \int |\hat{Q}|^2 \, dV. \quad (67)$$

Work against force $F'$ now leads to increase of potential energy of the plasma, $W^p$, of the plasma–vacuum surface, $W^s$, and of the vacuum, $W^v$. 
• Variables $\xi$ and $\hat{Q}$ have to satisfy **essential boundary conditions:**

1) $\xi$ regular on plasma volume $V$,  \hspace{1cm} (68)
\[ \nabla \times (\xi \times \hat{B}) = n \cdot \hat{Q} \] \hspace{1cm} (1st interface condition on $S$),  \hspace{1cm} (69)
2) $n \cdot \hat{Q} = 0$ \hspace{1cm} (on outer wall $\hat{W}$).  \hspace{1cm} (70)

• Note: Differential equations for $\hat{Q}$ and 2nd interface condition need not be imposed! They are absorbed in form of $W[\xi, \hat{Q}]$ and automatically satisfied upon minimization. For that reason 2nd interface condition \([63]\) is called **natural boundary condition**.

• Great simplification by assuming **incompressible perturbations**, $\nabla \cdot \xi = 0$:

\[
W_{\text{incl}}^p[\xi] = \frac{1}{2} \int \left[ |Q|^2 + j \cdot \xi^* \times Q - (\xi^* \cdot \nabla \Phi) \nabla \rho \cdot \xi \right] dV . \hspace{1cm} (71)
\]

Note: In equation of motion, one cannot simply put $\nabla \cdot \xi = 0$ and drop $-\gamma p \nabla \cdot \xi$ from pressure perturbation $\pi$, since that leads to overdetermined system of equations for 3 components of $\xi$. Consistent procedure: apply two limits $\gamma \to \infty$ and $\nabla \cdot \xi \to 0$ simultaneously such that Lagrangian perturbation $\pi_L \equiv -\gamma p \nabla \cdot \xi$ remains finite.
Application to Rayleigh–Taylor instability

- Apply extended energy principle to **gravitational instability of magnetized plasma supported from below by vacuum magnetic field**: Model problem for plasma confinement with clear separation of inner plasma and outer vacuum, and instabilities localized at interface (free-boundary or surface instabilities). Rayleigh–Taylor instability of magnetized plasmas involves the basic concepts of **interchange instability**, **magnetic shear stabilization**, and **wall stabilization**. These instabilities arise in wide class of astrophysical situations, e.g. **Parker instability** in galactic plasmas.

- Gravitational equilibrium in magnetized plasma:
  \[
  \rho = \rho_0, \quad \mathbf{B} = B_0 \mathbf{e}_z, \quad p = p_0 - \rho_0 g x, \quad (72)
  \]
  pressure balance at plasma–vacuum interface:
  \[
  p_0 + \frac{1}{2} B_0^2 = \frac{1}{2} \hat{B}_0^2, \quad (73)
  \]
  vacuum magnetic field:
  \[
  \hat{\mathbf{B}} = \hat{B}_0 (\sin \varphi \mathbf{e}_y + \cos \varphi \mathbf{e}_z). \quad (74)
  \]
• Insert equilibrium into $W_{\text{inc}}^p, W^s, W^v$, where jump in surface integral (66) gives driving term of the gravitational instability:

$$\mathbf{n} \cdot \left[ \nabla (p + \frac{1}{2} B^2) \right] = p' = -\rho_0 g.$$  \hspace{1cm} (75)

Potential energy $W[\xi, \hat{Q}]$ becomes:

$$W^p = \frac{1}{2} \int |\mathbf{Q}|^2 dV, \quad \mathbf{Q} \equiv \nabla \times (\xi \times \mathbf{B}), \quad \nabla \cdot \xi = 0,$$  \hspace{1cm} (76)

$$W^s = -\frac{1}{2} \rho_0 g \int |\mathbf{n} \cdot \xi|^2 dS,$$  \hspace{1cm} (77)

$$W^v = \frac{1}{2} \int |\hat{\mathbf{Q}}|^2 d\hat{V}, \quad \nabla \cdot \hat{\mathbf{Q}} = 0.$$  \hspace{1cm} (78)

Task: Minimize $W[\xi, \hat{Q}]$ for divergence-free trial functions $\xi$ and $\hat{Q}$ that satisfy the essential boundary conditions (68)–(70).

• Slab is translation symmetric in $y$ and $z$ ⇒ Fourier modes do not couple:

$$\xi = (\xi_x(x), \xi_y(x), \xi_z(x)) e^{i(k_y y + k_z z)}, \quad \text{similarly for} \quad \hat{Q}.$$  \hspace{1cm} (79)
• Eliminating $\xi_z$ from $W^p$, and $\hat{Q}_z$ from $W^v$, by using $\nabla \cdot \xi = 0$ and $\nabla \cdot \hat{Q} = 0$, yields 1D expressions:

$$W^p = \frac{1}{2} B_0^2 \int_0^a \left[ k_z^2(|\xi_x|^2 + |\xi_y|^2) + |\xi_x' + i k_y \xi_y|^2 \right] dx ,$$  \hspace{1cm} (80)

$$W^s = -\frac{1}{2} \rho_0 g |\xi_x(0)|^2 ,$$  \hspace{1cm} (81)

$$W^v = \frac{1}{2} \int_{-b}^0 \left[ |\hat{Q}_x|^2 + |\hat{Q}_y|^2 + \frac{1}{k_z^2} |\hat{Q}_x' + i k_y \hat{Q}_y|^2 \right] dx .$$  \hspace{1cm} (82)

• To be minimized subject to normalization that may be chosen freely for stability:

$$\xi_x(0) = \text{const} ,$$  \hspace{1cm} (83)

or full physical norm if we wish to obtain growth rate of instabilities:

$$I = \frac{1}{2} \rho_0 \int_0^a \left[ |\xi_x|^2 + |\xi_y|^2 + \frac{1}{k_z^2} |\xi_x' + i k_y \xi_y|^2 \right] dx .$$  \hspace{1cm} (84)

• Essential boundary conditions always need to be satisfied:

$$\xi_x(a) = 0 ,$$  \hspace{1cm} (85)

$$\hat{Q}_x(0) = i k_0 \cdot \hat{B} \xi_x(0) , \quad k_0 \equiv (0, k_y, k_z) ,$$  \hspace{1cm} (86)

$$\hat{Q}_x(-b) = 0 .$$  \hspace{1cm} (87)
Stability analysis

- Minimization with respect to $\xi_y$ and $\hat{Q}_y$ only involves minimization of $W^p$ and $W^v$:

$$W^p = \frac{1}{2} B_0^2 \int_0^a \left[ \frac{k_z^2}{k_0^2} \xi_x' + k_z^2 \xi_x + \left| \frac{k_y}{k_0} \xi_x' + i k_0 \xi_y \right|^2 \right] dx = \frac{1}{2} k_z^2 B_0^2 \int_0^a \left( \frac{1}{k_0^2} \xi_x'^2 + \xi_x^2 \right) dx,$$

$$W^v = \frac{1}{2} \int_{-b}^0 \left[ |\hat{Q}_x|^2 + \frac{1}{k_0^2} |\hat{Q}_x'|^2 \right] + \frac{1}{k_z^2} \left| \frac{k_y}{k_0} \hat{Q}_x' + i k_0 \hat{Q}_y \right|^2 dx = \frac{1}{2} \int_{-b}^0 \left( \frac{1}{k_0^2} |\hat{Q}_x'|^2 + |\hat{Q}_x|^2 \right) dx.$$

$\Rightarrow$ Determine $\xi_x(x)$ and $\hat{Q}_x(x)$, joined by 1st interface condition (86) at $x = 0$.

- Recall variational analysis: Minimization of quadratic form

$$W[\xi] = \frac{1}{2} \int_0^a \left( F \xi'^2 + G \xi^2 \right) dx = \frac{1}{2} \left[ F \xi' \right]_0^a - \frac{1}{2} \int_0^a \left[ (F \xi')' - G \xi \right] \xi dx \quad (88)$$

is effected by variation $\delta \xi(x)$ of the unknown function $\xi(x)$:

$$\delta W = \int_0^a \left( F \xi' \delta \xi' + G \xi \delta \xi \right) dx = \left[ F \xi' \delta \xi \right]_0^a - \int_0^a \left[ (F \xi')' - G \xi \right] \delta \xi dx = 0. \quad (89)$$

Since $\delta \xi = 0$ at boundaries, solution of Euler–Lagrange equation minimizes $W$:

$$(F \xi')' - G \xi = 0 \quad \Rightarrow \quad W_{\min} = \frac{1}{2} \left[ F \xi \xi' \right]_0^a = -\frac{1}{2} [F \xi \xi'](x = 0), \quad (90)$$

where we imposed upper wall BC $\xi(a) = 0$, appropriate for our application.
Minimization of integrals $W^p$ and $W^v$ yields following Euler–Lagrange equations, with solutions satisfying BCs on upper and lower walls:

$$\xi''_x - k_0^2 \xi_x = 0 \quad \Rightarrow \quad \xi_x = C \sinh [k_0(a - x)] ,$$  \hspace{1cm}(91)  

$$\hat{Q}''_x - k_0^2 \hat{Q}_x = 0 \quad \Rightarrow \quad \hat{Q}_x = i\hat{C} \sinh [k_0(x + b)] .$$

Modes are wave-like in horizontal, but evanescent in vertical direction.

$C$ and $\hat{C}$ determined by normalization (83) and 1st interface condition (86):

$$\hat{C} \sinh(k_0b) = k_0 \cdot \hat{B} \xi_x (0) = C k_0 \cdot \hat{B} \sinh(k_0a) .$$  \hspace{1cm}(92)  

Inserting solutions of Euler–Lagrange equations back into energy integrals, yields final expression for $W$ in terms of constant boundary contributions at $x = 0$:

$$W = \frac{k_0^2 B_0^2}{2k_0^2} \xi_x (0) \xi'_x (0) - \frac{1}{2} \rho_0 g \xi'^2_x (0) + \frac{1}{2k_0^2} |\hat{Q}_x (0) \hat{Q}'_x (0)|$$

$$= \frac{\xi'^2_x (0)}{2k_0 \tanh(k_0a)} \left[ (k_0 \cdot B)^2 - \rho_0 k_0 g \tanh(k_0a) + (k_0 \cdot \hat{B})^2 \frac{\tanh(k_0a)}{\tanh(k_0b)} \right] .$$  \hspace{1cm}(93)  

Expression inside square brackets corresponds to growth rate.
Growth rate

- With full norm \((84)\), we obtain \textit{dispersion equation of the Rayleigh–Taylor instability}:

\[
\omega^2 = \frac{W}{I} = \frac{1}{\rho_0} \left[ (k_0 \cdot B)^2 - \rho_0 k_0 g \tanh(k_0a) + (k_0 \cdot \hat{B})^2 \frac{\tanh(k_0a)}{\tanh(k_0b)} \right]. \tag{94}
\]

- \textit{Field line bending energies} \(\sim \frac{1}{2}(k_0 \cdot B)^2\) for plasma and \(\sim \frac{1}{2}(k_0 \cdot \hat{B})^2\) for vacuum, \textit{destabilizing gravitational energy} \(\sim -\frac{1}{2}\rho_0 k_0 g \tanh(k_0a)\) due to motion interface.

- Since \(B\) and \(\hat{B}\) not in same direction (\textit{magnetic shear} at plasma–vacuum interface), no \(k_0\) exists for which magnetic energies vanish \(\Rightarrow\) minimum stabilization when \(k_0\) on average perpendicular to field lines. Rayleigh–Taylor instability may then lead to \textit{interchange instability}: regions of high plasma pressure and vacuum magnetic field are interchanged.

- For dependence on magnitude of \(k_0\), exploit approximations of hyperbolic tangent:

\[
\tanh \kappa \equiv \frac{e^{\kappa} - e^{-\kappa}}{e^{\kappa} + e^{-\kappa}} \approx \begin{cases} 
1 & (\kappa \gg 1: \text{short wavelength}) \\
\kappa & (\kappa \ll 1: \text{long wavelength}) 
\end{cases} \tag{95}
\]

Short wavelengths \((k_0a, k_0b \gg 1)\): magnetic \(\gg\) gravitational term, system is stable. Long wavelengths \((k_0a \ll 1)\), and \(b/a \sim 1\): competition between three terms \((\sim k_0^2)\) so that effective \textit{wall stabilization} may be obtained.
Nonlinear evolution from numerical simulation

- Snapshot Rayleigh–Taylor instability for purely 2D hydrodynamic case: density contrast 10, (compressible) evolution.
- Shortest wavelengths grow fastest, ‘fingers’/‘spikes’ develop, shear flow instabilities at edges of falling high density pillars.
- Simulation with AMR-VAC: Versatile Advection Code, maintained by Gábor Tóth and Rony Keppens; Adaptive Mesh Refinement resolves the small scales.

Full nonlinear evolution (rthd.qt)