PLASMA PHYSICS Lecture Notes

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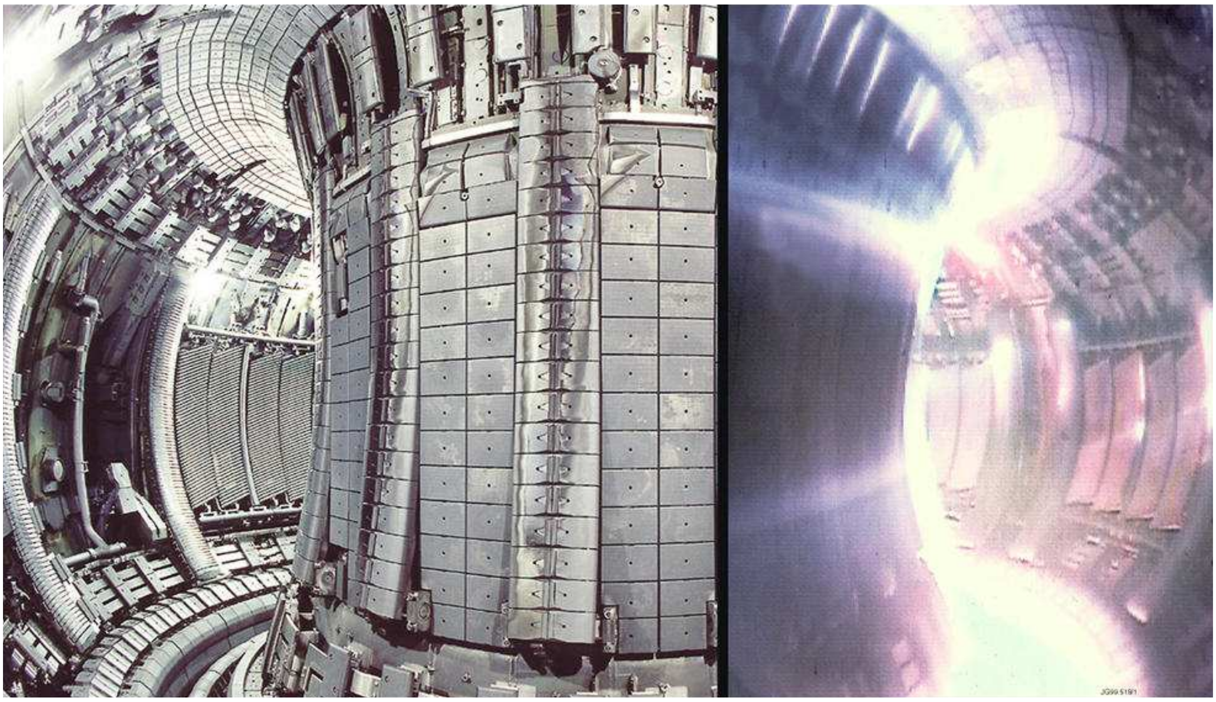

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1.1 *Introduction*

When a solid or liquid is heated, more and more molecules obtain an energy that exceeds the binding energy. The substance evaporates and enters the gas phase. If the temperature is increased yet further, the molecules will dissociate into atoms and ultimately some of their electrons become so energetic that the atoms ionize. At these high temperatures the gas becomes a mixture of negative electrons, positive ions, and neutral atoms. A gas which contains so many charged particles that they dominate its behaviour, is called a *plasma*.

More than 99% of all visible matter in the universe is in the plasma state. Plasma is sometimes called the fourth state of matter along with the solid, liquid and gaseous states. One difference from the other states is the absence of a clear phase transition. Instead, there is a continuous transition from gases with neutral atoms to plasmas with ionized atoms, which is determined by a dissociation equation. Thus, the transition between a gas and a plasma is essentially a chemical equilibrium which shifts from the gas to the plasma side with increasing temperature.

Let us assume for simplicity that the atoms can exist in precisely two states: the ground state and the ionized state. The ionization energy is E_i. The ionization reaction rate strongly increases with temperature T whereas the recombination reactions depend only weakly on T. The thermodynamical equilibrium between ionization and recombination reactions at temperature T is described by the Saha equation,

$$\frac{n_e n_i}{n_0} = \frac{\pi_e \pi_i}{\pi_0} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp(-E_i/kT), \tag{1}$$

where $n_e = n_i$ are the electron and ion densities, n_0 the density of the neutrals, h Planck's constant, k Boltzmann's constant, m_e the electron mass, and $\pi_{e,i,0}$ the statistical weights of the electrons ($\pi_e = 2$), ions, and neutrals.

For hydrogen, which has $\pi_i = 2$, $\pi_0 = 4$, and an ionization-energy of $E_i = 13.6$ eV (1 eV corresponds to a temperature of 1.16×10^4K), the Saha equation reads

$$\frac{n_e n_i}{n_0} = 2.4 \times 10^{21}\, T^{3/2} \exp(-E_i/kT)\ \mathrm{m}^{-3}, \tag{2}$$

where the temperature T is in K. Figure 1 shows the ionization degree $n_i/(n_0 + n_i)$ of hydrogen gas, plotted as a function of the gas temperature T at a constant total pressure $p = (n_0 + n_e + n_i)kT$. The graph shows that the gas is already fully ionized at thermal energies well below the ionization-energy of 13.6 eV. At about 1/10 of the ionization energy the majority of particles are ionized. Already at lower temperatures the electrically charged components of a partially ionized gas can dominate the behaviour of the gas. A characteristic property of neutral

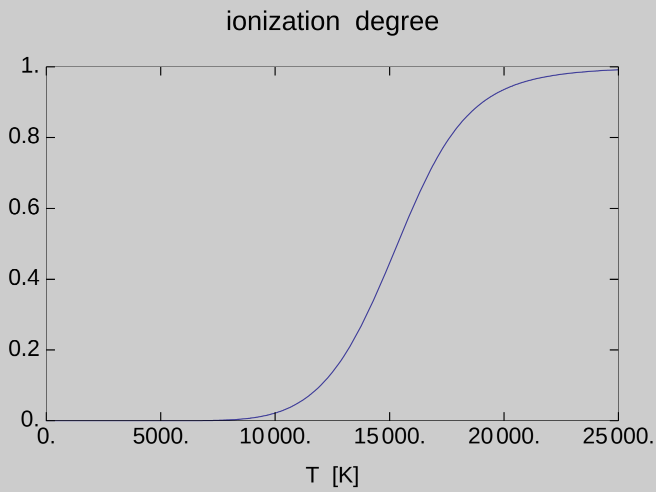


Figure 1: Temperature dependence of the ionization degree $n_i/(n_0 + n_i)$ of hydrogen at a constant pressure of 10^5 Pa. The ionization energy is 13.6 eV (equivalent to 157000 K).

gases is the short range of the intermolecular forces, shorter than the average distance between the particles. Therefore, in kinetic gas theory, molecules are considered to be freely moving particles, which intermittently change velocity and direction due to collisions. The situation in an ionized gas is entirely different. Here, the interactions between the particles are given by the Coulomb potential, which only decreases with distance as r^{-1}.

The number of particles enclosed in a fixed solid angle increases with r^2, so that the interaction force does not decrease with the distance. The charges are in motion. Hence, magnetic interactions play a role as well. Also the

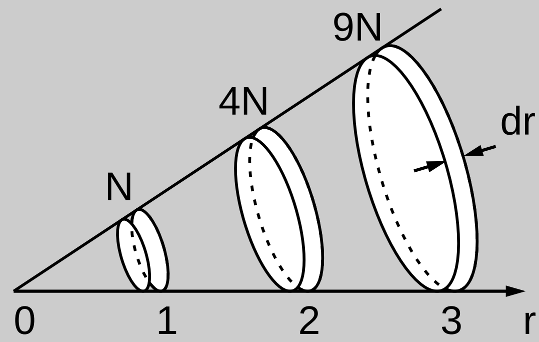


Figure 2: The number of particles enclosed in a fixed solid angle increases as r^2.

magnetic forces have a long range. Thus, there are forces between distant parts of the plasma. These interactions can be coherent, involving collective behaviour of many particles that has no analogy in ordinary gases.

The moving charges generate electric fields and currents which, via Maxwell's equations, generate magnetic fields. The general meaning of the word **plasma** is a collection of charged particles that is sufficiently dense so that their space charge effects lead to strongly collective behaviour.

1.2 *Ideal Plasma, the Landau length*

Consider a gas consisting of particles with charge e and volume density n. The gas is a many-particle system, which means that the average distance $\langle r\rangle = n^{-1/3}$ between the particles is small compared with the characteristic size L of the system, $n^{-1/3} \ll L$.

An **ideal plasma** is defined as a plasma in which the interaction energy between the particles is small compared to the kinetic energy of the particles,

$$\frac{e^2}{4\pi\epsilon_0\langle r\rangle} = \frac{e^2\, n^{1/3}}{4\pi\epsilon_0} \ll \tfrac{3}{2}kT. \tag{3}$$

This is analogous to the definition of an ideal gas, where van der Waals interactions are negligible.

The Landau length λ_L is defined as the distance of closest approach between particles with the average kinetic energy $\frac{3}{2}kT$. At the point of closest approach this energy is equal to the Coulomb potential ϕ,

$$\tfrac{3}{2}kT = e\phi = \frac{e^2}{4\pi\epsilon_0\lambda_L},$$

Therefore, an ideal plasma is characterized by an average interparticle distance that is large compared to the distance of closest approach,

$$\langle r\rangle = n^{-1/3} \gg \lambda_L = \frac{e^2}{6\pi\epsilon_0 kT}. \tag{4}$$

1.3 *Quasi-neutrality*

The above discussion also applies to plasmas that posess a net electrical charge. However, such plasmas are not very common. There is under most circumstances no static force that can balance the strong repulsive electrostatic forces (except the gravitational force, which is usually negligible because of the small mass density of the plasma).

A simple estimate shows that a small charge separation causes a large restoring electric field. A relative charge density surplus of $\Delta n/n_e = 10^{-3}$ in a homogeneous sphere of radius r generates, according to Poisson's law $\nabla\cdot\boldsymbol{E} = \epsilon_0^{-1}e\Delta n$, at the surface of the sphere a radial electric field

$$E_r = \frac{e}{3\epsilon_0}r\Delta n \simeq 6\;10^{-12} r n_e \text{ V/m}.$$

In nuclear fusion experiments a typical plasma size and density are $r \sim$ 1m and $n_e \sim 10^{20}\,\text{m}^{-3}$. A charge surplus of just 0.1% gives rise to a gigantic electric field of $E_r = 6\;10^8$ V/m! Thus, non-neutral plasmas are characterized by large internal electric fields, and can exist only in non-stationary situations far from thermodynamic equilibrium, as a transient phenomenon, or in situations with strong plasma flow. If a plasma contains positive as well as negative charged particles, it has a strong tendency towards electric neutrality.

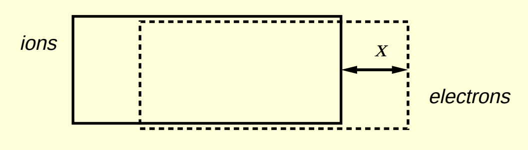


Figure 3: Schematic illustration of charge separation in a plasma.

Not only is the charge density in a plasma generally very small, also the distance of charge separation tends to be very limited. Consider a non-zero charge density caused by a small separation of the electrons from the ions (see figure 3) over a distance x. Application of Poisson's equation $\nabla \cdot \boldsymbol{E} = \rho_E/\epsilon_0 \approx E/x$ yields

$$E \approx \frac{n_e e\, x}{\epsilon_0}. \tag{5}$$

A fully ionized plasma at 5 eV and atmospheric pressure has an electron density of $n_e = 6 \cdot 10^{22}\mathrm{m}^{-3}$. A charge separation of 1 mm would lead to a very strong electric field $E = 10^{12}$ V/m and an electric potential $\phi = 5 \cdot 10^8$ V over 1 m. One sees that finite electric fields can be generated by very small deviations of the densities from $\sum_j Z_j n_j = n_e$. The index j labels ion species with positive electric charge eZ_j per ion and density n_j. Therefore one can often apply the **plasma approximation**: one assumes exact neutrality, $n_e = \sum_j Z_j n_j$, but does not exclude sources of the electric field, $\nabla \cdot \boldsymbol{E} \neq 0$. It will be shown in Chapter 6 how the electric field in a quasi-neutral plasma is determined via Ohm's law instead of the Poisson equation.

1.4 *The plasma frequency*

The strong restoring electric field associated with deviations from electric neutrality causes a harmonic oscillation. Because of their small mass, the electrons form the most mobile component of the plasma: they respond much quicker to the electric field than the ions. Consider again the charge separation over a distance x shown in figure 3. In the absense of a magnetic field, electrons react to the electric field of equation (5) with

$$m_e \frac{d^2 x}{dt^2} = -e\, E = -\frac{n_e e^2\, x}{\epsilon_0}.$$

The solution of this equation is an oscillation with the characteristic *plasma frequency*

$$\omega_p = \left(\frac{n_e e^2}{\epsilon_0\, m_e} \right)^{1/2}. \tag{6}$$

This sets the timescale ω_p^{-1} on which the electron gas responds to electric fields. The ions do not participate in these oscillations because of their large mass. The plasma frequency is temperature independent. For typical fusion plasma densities ($10^{20}\mathrm{m}^{-3}$, see Fig. 4) one obtains $\nu_p \approx 100$ GHz. Above this frequency electromagnetic waves can propagate, while for $\omega < \omega_p$ the electrons screen the wave. In the latter case the electromagnetic waves can penetrate only to a depth

$$d_e \equiv \frac{c}{\omega_{pe}} = 5.31 \cdot 10^6\, n_e^{-1/2}\,\mathrm{m}. \tag{7}$$

into the plasma. The length d_e is called the inertial skin depth, and is similar to the skin depth for the penetration of electromagnetic waves into a metal (where the conducting electrons form the "plasma"). In Chapter 5 the electromagnetic wave propagation and shielding will be treated in more detail, clarifying the collective oscillations of the electrons and the averaged electric and magnetic fields.

1.5 *Debye shielding*

While a plasma tends towards quasi-neutrality, the random thermal motion of the particles create small fluctuations of the charge density all the time. However, such charges cannot give rise to differences in potential energy of the

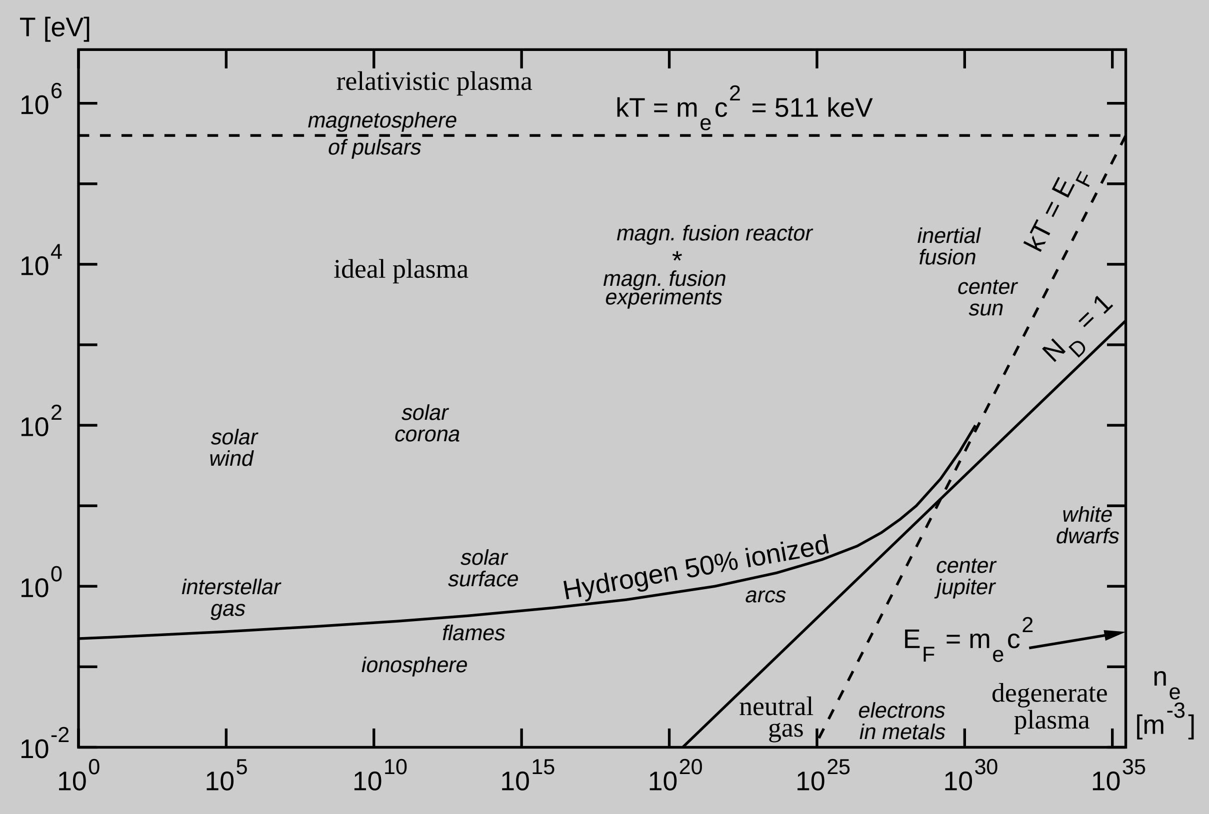


Figure 4: Typical temperatures and densities of astrophysical and laboratory plasmas.

particles in different locations that are much larger than the thermal kinetic energy. In a plasma the thermal energy density of the electrons per degree of freedom is $\frac{1}{2}n_e\,kT$, and that is therefore the level of the potential energy fluctuations that can arise due to thermal motion. Let us assume that this potential energy results from a separation of charge over a length x (see figure 3 and equation (5)), i.e., $E_{pot} = \frac{1}{2}\epsilon_0\,E^2 = \frac{1}{2}\epsilon_0\,(n_e ex/\epsilon_0)^2$. One sees that substantial separation of charge can only occur over lengths to $x < \lambda_D$, where

$$\lambda_D = \left(\frac{\epsilon_0 kT}{e^2 n}\right)^{1/2}, \tag{8}$$

is called the Debye-length, after P. Debye who applied it to the charged particles in an electrolyte.

Note, that the thermal electron velocity $v_{th} = \sqrt{kT/m}$ is found from (6) and (8) to be

$$v_{th,e} = \lambda_D\,\omega_p.$$

Therefore, $1/\omega_p$ is the time a thermal electron needs in order to pass the Debye length. One important consequence is that, as mentioned before, electro-magnetic waves with frequencies below the plasma frequency cannot propagate in plasmas since their electric fields are screened by the Debye shielding. Instead the waves are reflected. For larger frequencies the inertia of the electrons allows the propagation of the waves. This is the reason why metals reflect visible light but not x-rays.

In terms of the charge density $\rho_E = e(\sum_j Z_j n_j - n_e)$, Poisson's equation shows that the potential difference $\Delta\phi$ over a distance x is

$$\Delta\phi \sim \frac{x^2}{2}\nabla^2\phi = -\frac{e}{\epsilon_0}\frac{x^2}{2}(\sum n_i Z_i - n_e), \tag{9}$$

For the electrons, the inequality $e\Delta\phi \le \frac{1}{2}kT_e$ implies that

$$\frac{\sum n_i Z_i - n_e}{n_e} \le \frac{\lambda_D^2}{x^2}. \tag{10}$$

The Debye length λ_D is the fundamental length scale that characterizes a plasma. Equation (10) implies that quasi-neutrality holds at length scales that are large compared to the Debye length, $L \gg \lambda_D$. At distances smaller than λ_D a thermal plasma does exhibit charge variations.

We shall now give a more detailed derivation of the Debye shielding of electric charges in the plasma which takes into account both the electrons and the ions. In fact, each individual charged particle is a —very localized— deviation of neutrality, which is shielded by the surrounding charges. The detailed mechanism is that around each individual charged particles, the surrounding charges adjust to the electric field of the central charge, effectively screening this electric field at distances exceeding λ_D. To see how the different charged particle species co-operate in this effect, consider the electric potential of a charge q located in the origin $\boldsymbol{x} = 0$. Including all other particles with densities n_j, the potential ϕ satisfies

$$\nabla^2\phi = -\frac{q}{\epsilon_0}\delta(\boldsymbol{x}) - \frac{1}{\epsilon_0}\sum_j Z_j e n_j, \tag{11}$$

where the sum is now over all particle species $(Z_e = -1)$. The boundary conditions are

$$\phi(r \to 0) \to \frac{q}{4\pi\epsilon_0 r}, \qquad \phi(r \to \infty) \to 0. \tag{12}$$

In a thermal plasma, the particle densities are given by Boltzmann distributions,

$$n_j(r) = \bar{n}_j \exp(-Z_j e\phi/kT). \tag{13}$$

with $n_j(r \to \infty) = \bar{n}_j$. The condition of charge neutrality at infinity is $\Sigma_j Z_j e \bar{n}_j = 0$. Let us assume that the plasma is ideal so that the electric potential is relatively weak, $Ze\phi/kT \ll 1$. Then in (13) we may linearize the Boltzmann factors $n_j \approx \bar{n}_j(1 - Z_j e\phi/kT)$. Thus we find from (11)

$$\nabla^2\phi = -\frac{q}{\epsilon_0}\delta(\boldsymbol{x}) + \frac{1}{{\lambda_D}^2}\phi, \tag{14}$$

where, taking into account all particle species, the Debye length is given by

$$\lambda_D^2 = \frac{\epsilon_0 kT}{\sum Z_j^2 e^2 \bar{n}_j} \tag{15}$$

For spherical symmetric functions $\phi(r)$ one has

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{d\phi}{dr}\Big),$$

so that spherically symmetric solutions of (14) that satisfy the boundary conditions have the form of a Yukawa potential,

$$\phi(r) = \frac{q}{4\pi\epsilon_0}\frac{\exp(-r/\lambda_D)}{r}. \tag{16}$$

The charge density is

$$\rho_E = \sum_j Z_j e n_j = -\frac{q}{4\pi\lambda_D^2}\frac{\exp(-r/\lambda_D)}{r}. \tag{17}$$

Around a charge q an extended charge cloud of opposite sign forms, which decreases exponentially with distance and screens the electric field of q at large distances. The total charge of the cloud is $-q$ and its characteristic size is the Debye length λ_D

Thus, on distances shorter than λ_D the charged particles interact via their microscopic Coulomb fields. At distances larger than λ_D this field is screened and the interaction between plasma elements is determined by the smoothed-out, averaged, electric and magnetic fields to which all plasma particles contribute.

1.6 *The plasma parameter*

The above derivation is justified only if the number of particles involved in the screening effect is large,

$$N_D \gg 1, \tag{18}$$

where we have defined the **plasma parameter**

$$N_D \equiv \tfrac{4}{3}\pi n \lambda_D^3, \tag{19}$$

as the number of particles in the so-called Debye sphere. If we consider, for simplicity, only the electrons, the Debye length is

$$\lambda_D = \Big(\frac{\epsilon_0 kT}{e^2 n}\Big)^{1/2}.$$

The plasma parameter can be expressed in terms of the Landau length λ_L and the average particle distance $\langle r \rangle = n^{-1/3}$,

$$N_D = \tfrac{2}{9}\frac{\lambda_D}{\lambda_L} = \tfrac{2}{9}(6\pi)^{-1/2}\Big(\frac{\langle r \rangle}{\lambda_L}\Big)^{3/2}.$$

Hence requirement (18) of having many particles in the Debye sphere is equivalent to condition (4) that the mean distance between particles is much larger than the shortest distance during Coulomb interactions and equivalent to condition (3) for an ideal plasma.

Let us inspect once more the result of the previous section that each charge is surrounded by a charge cloud (17) with potential (16). The average potential ϕ_j of a particle of species j due to all other charges is

$$\phi_j = \lim_{r\to 0} \frac{Z_j e}{4\pi\epsilon_0 r}[\exp(-r/\lambda_D) - 1] = \frac{-Z_j e}{4\pi\epsilon_0 \lambda_D}. \tag{20}$$

The averaged potential energy density due to the Coulomb interactions is therefore

$$W = \sum_{i\neq j} \frac{Z_i Z_j e^2}{4\pi\epsilon_0 |\boldsymbol{x}_i - \boldsymbol{x}_j|} = \tfrac{1}{2}\sum_j Z_j e n_j \phi_j = -\frac{1}{8\pi}\frac{\sum_j Z_j^2 e^2 n_j}{\epsilon_0 \lambda_D} = -\frac{1}{8\pi}\frac{kT}{\lambda_D^3}. \tag{21}$$

One sees that in quasi-neutral plasmas the average potential energy density is much smaller than the kinetic energy density $K = \sum_j \frac{3}{2} n_j kT$,

$$\frac{W}{K} \sim \frac{1}{N_d} \ll 1. \tag{22}$$

We see once more that, if there are many particles in a Debye-sphere, one has an ideal plasma.

If we take $Z_j = 1$, the length λ_D can be computed from

$$\begin{aligned} \lambda_D &= 69\Big(\frac{T}{n}\Big)^{1/2}\,\mathrm{m} \qquad (T \text{ in K}) \\ &= 7434\Big(\frac{kT}{n}\Big)^{1/2}\,\mathrm{m}, \qquad (kT \text{ in eV}). \end{aligned} \tag{23}$$

For a thermonuclear plasma with $kT = 20\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$ one finds $\lambda_D \simeq 10^{-4}\,\mathrm{m}$ and $N_d \simeq 4\;10^8$. Of course, the Debye length is only a meaningful scale in a plasma if it is small compared to the typical length scale of the plasma: $L \gg \lambda_D$. Concluding, an ideal, quasi-neutral plasma is defined by the ratios

$$n^{-1/3} \ll \lambda_D \ll L \tag{24}$$

between the characteristic length scales.

The microscopic interactions between particles within λ_D can be described by a “collision frequency” that can be estimated heuristically as follows. The distance of closest approach of two charges is $\lambda_L \sim e^2/\epsilon_0 kT$. If we take this distance as representative for the collision cross-section $\sigma \sim \lambda_L^2$ and use the average (thermal) electron velocity $v_{th,e} \sim (kT/m_e)^{1/2}$ then the collision frequency ν_{ce} for an electron is

$$\nu_{ce} = n\langle \sigma v \rangle \sim n\lambda_L^2 (kT/m_e)^{1/2} \sim \frac{ne^4}{\epsilon_0^2 (kT)^{3/2} m_e^{1/2}}. \tag{25}$$

From Eqs. (6) and (25) one sees that

$$\frac{\nu_{ce}}{\omega_p} \sim \frac{1}{N_d} \ll 1. \tag{26}$$

This means that the dynamical time scale ω_p^{-1} is much shorter than the typical timescale $\tau_e = \nu_c^{-1}$ on which thermal energy can be redistributed. Also from this viewpoint one sees that collective interactions dominate over binary interactions between particles. In terms of the mean free path $\lambda_{\rm mfp} = \nu_{ce}^{-1} v_{th,e} = \nu_{ce}^{-1}(kT_e/m_e)^{1/2}$ one can rewrite (26) as

$$\frac{\lambda_D}{\lambda_{\rm mfp}} \sim \frac{1}{N_d} \ll 1, \tag{27}$$

where we have used $\lambda_D = v_{th,e}/\omega_p$. Eqs. (25) and (26) express that collisions are infrequent. This explains why thermodynamic equilibrium is a rare situation in laboratory and space plasmas. There are many situations where $\lambda_{\rm mfp}$ even exceeds the linear size of the plasma. In such situations the plasma paramaters are not local quantities but depend on the global properties of the entire plasma, such as its shape.

The relative sizes of the various length scales that we have introduced in this chapter can be summarized as follows, ($\langle r \rangle = n^{-1/3}$):

$$\left(\frac{\lambda_L}{\langle r \rangle}\right)^{3/2} \sim \left(\frac{\langle r \rangle}{\lambda_D}\right)^3 \sim \frac{\lambda_D}{\lambda_{\rm mfp}} \sim \frac{1}{N_d} \ll 1. \tag{28}$$

In the limit $N_d \to \infty$ all characteristic quantities that describe the fact that the plasma consists of discrete particles vanish. The local, microscopic fields are unimportant in this limit and the plasma behaviour is fully determined by average electromagnetic fields.

	n m^{-3}	T eV	ω_p sec^{-1}	λ_D m	$n\lambda_D^3$	τ_e^{-1} sec^{-1}
Interstellar gas	10^6	1	$6\ 10^4$	7	$4\ 10^8$	$7\ 10^{-5}$
Gaseous nebula	10^9	1	$2\ 10^6$	$2\ 10^{-1}$	$8\ 10^6$	$6\ 10^{-2}$
Solar corona	10^{15}	10^2	$2\ 10^9$	$2\ 10^{-3}$	$8\ 10^6$	50
Diffuse hot plasma	10^{18}	10^2	$6\ 10^{10}$	$7\ 10^{-5}$	$4\ 10^5$	$4\ 10^4$
Solar atmosphere gas discharge	10^{20}	1	$6\ 10^{11}$	$7\ 10^{-7}$	40	$2\ 10^9$
Warm plasma	10^{20}	10	$6\ 10^{11}$	$2\ 10^{-6}$	$8\ 10^2$	$9\ 10^7$
Hot plasma	10^{20}	10^2	$6\ 10^{11}$	$7\ 10^{-6}$	$4\ 10^4$	$4\ 10^6$
Thermonuclear plasma	10^{21}	10^4	$2\ 10^{12}$	$2\ 10^{-5}$	$8\ 10^6$	$5\ 10^4$
Dense hot plasma	10^{24}	10^2	$6\ 10^{13}$	$7\ 10^{-8}$	$4\ 10^2$	$2\ 10^{10}$
Laser plasma	10^{26}	10^2	$6\ 10^{14}$	$7\ 10^{-9}$	40	$2\ 10^{12}$

(Source: D.L. Book, NRL Plasma Formulary, 1994.)

A number of characteristic parameters of space and laboratory plasmas are listed in the above table.

2 Coulomb collisions

2.1 *Collision times*

In a fully ionized plasma, the transfer of energy and momentum between the individual particles and the ensuing diffusion and transport processes are caused by Coulomb collisions between the particles. Here we use the term "collisions" for all interactions between the particles, after the "smoothed" electric and mangetic fields are subtracted. Since the average distance between the particles is small compared to the long range of the Coulomb force (the number of particles in a Debye sphere is large), every charged particle undergoes interactions with a very large number of particles simultaneously. This is a complicated many-particle system. We will approximate this system with a two-particle model: the many-particles interaction is replaced by a sequency of instantaneous two-particle interactions. Every continuous two-particle interaction is replaced by an instantaneous collision between the two particles, with exactly the same transfer of energy and momentum as in the continuous two-particle interaction.

The collision process is most easily described in the center-of-mass sytem of the two particles. The geometry is scetched in the following figure. The relation between the scattering angle χ, the impact parameter b, and the

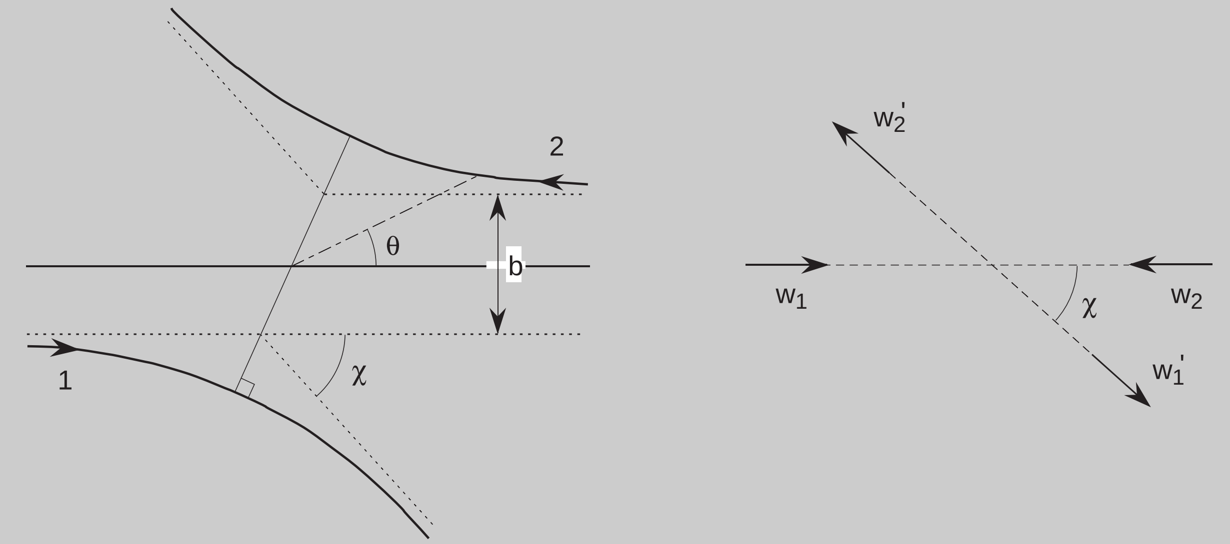


Figure 5: The geometry of a two-particle collision in the center-of-mass frame. The distance between the particles is $r(t)$. The consequence of the collision is a rotation of the relative velocity vector $\boldsymbol{w}$ over an angle χ, the scattering angle.

relative velocity $w_0 = |\boldsymbol{w}_1 - \boldsymbol{w}_2|_0$ long after the collision of two particles (with masses m_1, m_2 and charges q_1, q_2) is (see Appendix)

$$\cot \tfrac{1}{2}\chi = \frac{b}{b_0}, \tag{29}$$

where

$$b_0 = \frac{q_1 q_2}{4\pi\epsilon_0 \mu w_0^2} \tag{30}$$

is the impact parameter for $90°$ deflection, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. According to (30), b_0 depends on the relative velocities of the colliding particles. For thermal particles we take $\mu w_0^2 \approx 3kT$. Then we find $b_0 = \frac{1}{2}\lambda_L$ and $n^{1/3} b_0 \sim \left(n\lambda_D^3\right)^{-2/3} \ll 1$. One sees that b_0 is small compared to the distance between the particles. The differential scattering cross section $\sigma(\chi)$ and the impact parameter b are related via

$$b\,db = \sigma(\chi) \sin\chi \, d\chi. \tag{31}$$

From (29) and (31) follows Rutherford's formula,

$$\sigma(\chi) = \frac{b_0^2}{4 \sin^4 \frac{1}{2}\chi}. \tag{32}$$

The cross section for a scattering angle larger than 90° is therefore

$$\sigma_0 = 2\pi \int_{\pi/2}^{\pi} \sigma(\chi) \sin\chi \, d\chi = \pi b_0^2. \tag{33}$$

The initial relative velocity is w_0. After scattering, the velocity component in the same direction is $w_0(1 - \cos\chi)$. Integrating over all scattering directions, we find the effective scattering cross section for momentum transfer to be

$$\sigma_1 = 2\pi \int (1 - \cos\chi)\sigma(\chi) \sin\chi \, d\chi \tag{34}$$

Note that the scattering cross section (32) is strongly peaked for small scattering angles, i.e., for large values of the impact parameter. Therefore, the integral (34) is dominated by "soft" collisions with small scattering angles. This is typical for a long range interaction such as the Coulomb force.

The Debye shielding limits the range of the Coulomb interaction. Therefore, large impact parameters $b > \lambda_D$ do not contribute to the scattering cross section.

Writing the integral (34) in terms of the impact parameter, one finds

$$\sigma_1 = 4\pi \int_0^{\lambda_D} db \frac{b}{1 + b^2/b_0^2} = 4\pi b_0^2 \ln\Lambda \tag{35}$$

where $\Lambda = \lambda_D/b_0 \sim N_D \gg 1$. One sees that the cross section σ_1 is larger than the cross section σ_0 for large angle collisions by a factor $4 \ln\Lambda$.

Depending on the impact parameter, Coulomb interactions in a plasma can be characterized as follows:

1. $b < b_0 \ll n^{-1/3} \ll \lambda_D$ hard, binary collisions with large scattering angles.
2. $b_0 \ll b \ll n^{-1/3} \ll \lambda_D$ soft, binary collisions with small scattering angles.
3. $n^{-1/3} \ll b < \lambda_D$ many simultaneous soft collisions with small scattering angles. The Coulomb potential is slightly modified by Debye shielding.
4. $\lambda_D < b$ the Coulomb potential is screened and the interaction takes place via the average E and B fields of all particles collectively.

Collisions in a plasma are responsible for a number of relaxation processes, such as the relaxation of the particle energy distribution towards a thermal equilibrium. The characteristic relaxation times for coulomb collisions depend on the velocity distributions of the particles since σ is a strongly decreasing function of w_0. Therefore, an accurate calculation of relaxation processes is kinetic, involving integrals over the particle velocity distributions. Such calculations are too detailed for the present discussion. Here we limit ourselves to a number of results for plasmas that are close to thermal equilibrium. The details are given in the Appendix.

For a plasma consisting of electrons with temperature T_e, and of ions with temperature T_i (in keV), charge Ze, and density n_i (in m^{-3}, the following characteristic times can be given. The electron collision time is

$$\tau_e = \frac{12\pi^{3/2}}{2^{1/2}} \frac{\epsilon_0^2 m_e^{1/2} T_e^{3/2}}{n_i Z^2 e^4 \ln\Lambda_e} = 1.09\,10^{16} \frac{T_e^{3/2}}{n_i Z^2 \ln\Lambda_e} \text{ sec.} \tag{36}$$

It is the time it takes to deflect thermal electrons via collisions with the ions. For $Z = 1$ electron-electron collisions have comparable effects, but for larger Z the collisions with the ions are more important.

The ion-ion collision time is a factor $(m_i/m_e)^{1/2}$ longer,

$$\tau_i = 12\pi^{3/2} \frac{\epsilon_0^2 m_i^{1/2} T_i^{3/2}}{n_i Z^4 e^4 \ln\Lambda_i} = 6.60\,10^{17} \left(\frac{m_i}{m_p}\right)^{1/2} \frac{T_i^{3/2}}{n_i Z^4 \ln\Lambda_i} \text{ sec,} \tag{37}$$

where m_p is the proton mass. The energy exchange time between electrons and ions is

$$\tau_{ei} = \frac{m_i}{2m_e}\tau_e = 0.99\,10^{19} \frac{m_i}{m_p} \frac{T_e^{3/2}}{n_i Z^2 \ln\Lambda_e} \text{ sec.} \tag{38}$$

Energy transfer between electrons and ions is much slower than momentum transfer because $m_e \ll m_i$.

In expressions (36)–(38), $\ln\Lambda$ is the so-called Coulomb logarithm, which measures how much stronger the combined effect of many small-angle collisions is compared with large-angle scattering with the nearest particles. The following plug-in formulas take the density in m^{-3} and the temperature in keV. In the case of *electron-ion* collisions ($T_e \geq 10\,\mathrm{eV}$)

$$\ln\Lambda_e = 15.2 - \tfrac{1}{2}\ln(10^{-20}n_e) + \ln T_e, \tag{39}$$

in the case of *electron-electron* collisions ($T_e \geq 10\ eV$)

$$\ln\Lambda_e = 14.9 - \tfrac{1}{2}\ln(10^{-20}n_e) + \ln T_e, \tag{40}$$

and for *ion-ion* collisions ($T_i \leq 10m_i/m_p\ keV$)

$$\ln\Lambda_i = 17.3 - \tfrac{1}{2}\ln(10^{-20}n_e) + \tfrac{3}{2}\ln T_i, \tag{41}$$

Table: Collision times for electrons and deuterons

n [m^{-3}]	T [keV]	τ_e	τ_i	$\tau_{i,e}$
10^{19}	0.1	2.4 μs	0.2 ms	4.4 ms
10^{19}	1	67 μs	5 ms	120 ms
10^{19}	10	1900 μs	130 ms	3400 ms
10^{20}	0.1	0.27 μs	0.021 ms	0.49 ms
10^{20}	1	7.2 μs	0.54 ms	13 ms
10^{20}	10	200 μs	14 ms	370 ms

Plasma processes at typical time scales that are short compared to the collision times (36)-(38) can be described collisionless to first approximation. The characteristic collision times increase with the plasma temperature as $T^{3/2}$. This is why high temperature plasmas behave as a practically collisionless medium.

2.2 *Quantum effects*

In most plasmas quantum effects are unimportant because the DeBroglie wavelength $\lambda_{\mathrm{dB}} = \hbar/mv$ is small compared to the average particle distance $\langle r\rangle = n^{-1/3}$. However, the classical description of the collision process must be corrected for quantum effects if they exceed the impact parameter for Coulomb scattering, $\lambda_{\mathrm{dB}} > b_0$. This is the case for electron energies $kT_e > 10\,\mathrm{eV}$. For protons, the critical energy is $\approx 10\,\mathrm{keV}$. These corrections have been included in the expressions (39) and (40).

What does this mean for the criterion for an ideal plasma? Just like the ideal plasma criterion (3) can be written as a lower bound on the closest distance, $\langle r\rangle \gg \lambda_L$ (4), can the quantum limit be expressed as a lower bound on the closest distance, $\langle r\rangle > \lambda_{\mathrm{dB}}$. This means that in an ideal plasma quantum effects do not play a role if $\lambda_L > \lambda_{\mathrm{dB}}$. This criterion can be converted back into an inequality for the average particle energy,

$$kT \gg \frac{e^2 m_e}{(4\pi\epsilon_0)^2\hbar^2} = 2E_I.$$

Thus, in an ideal plasma the thermal energy is large compared with the ionization energy E_I.
Typical fusion plasma parameters are

- density $n_e = 10^{20}\mathrm{m}^{-3}$
- Debye length is $\lambda_D = 75\,\mu\mathrm{m}$
- Landau length $\lambda_L = 1\cdot10^{-13}\,\mathrm{m}$,
- thermal de Broglie length $\lambda_{\mathrm{dB}} = 3\cdot10^{-12}\mathrm{m}$
- plasma parameter $N_D = 2\cdot10^8$.

For nonideal plasmas ($\langle r \rangle < \lambda_L$) there is a high-density limit $\langle r \rangle = \lambda_{\rm dB}$ above which quantum effects are important. For higher densities, the plasma becomes degenerate. Examples are the free electrons in a metal, and the plasma of a white dwarf star. This regime can be characterized by the average particle energy being exceeded by the Fermi energy E_F,

$$\frac{3}{2}\,kT \ll E_F = \frac{\hbar^2}{2m_e}\,(3\pi^2 n)^{2/3}.$$

Under these conditions, to the right of the dashed line in Figure 4, the normal Boltzmann statistics is replaced by Fermi-Dirac statistics.

2.3 *Electric resistivity*

In a plasma, one of the most important effects of collisions between ions and electrons is the electric resistivity. As we shall see in Chapter 7, this resistivity has a profound influence on changin magnetic fields in a plasma. For now, we discuss the electric resistivity in the absence of a magnetic field. In a continuum, Ohm's law relates the electric field vE to the current density $\boldsymbol{J}$ via the resistivity η (units: Ωm),

$$\boldsymbol{E} = \eta \boldsymbol{J} \tag{42}$$

We shall now derive how this resistivity follows from the friction (collisions) between the electrons and ions. After all, an electric current implies different average velocities for the electrons and ions. Assuming an average electron velocity $\boldsymbol{u}_e \gg \boldsymbol{u}_i$ and electron density n_e, the current density is $\boldsymbol{J} \approx -en_e\boldsymbol{u}_e$. Ohm's law gives the following force per unit volume on the electrons in the Electric field:

$$-e\,n_e\,\boldsymbol{E} = -e\,n_e\,\eta\,\boldsymbol{J}, \tag{43}$$

This increase is balanced by a friction force (per unit volume) due to collisions of electrons and ions, proportional to the collision frequency,

$$-\frac{n_e m_e\,\boldsymbol{u}_e}{\tau_e} = \frac{m_e\,\boldsymbol{J}}{e\tau_e}. \tag{44}$$

Balancing (42) and (43) gives

$$\eta = \frac{m_e}{n_e e^2 \tau_e} = \frac{2^{1/2}}{12\pi^{3/2}}\frac{m_e^{1/2}\,Ze^2\ln\Lambda_e}{\epsilon_0^2\,T_e^{3/2}}. \tag{45}$$

Note that the plasma resistivity strongly decreases with temperature. For $T_e = 1.4\,$keV the plasma resistivity is comparable to copper ($1.8 \cdot 10^{-8}\,\Omega$m). Since $\eta \sim T_e^{-3/2}$, fusion plasmas with $T \approx 10\,$keV have an order of magnitude smaller resistivity. The dependence on the density is much weaker, only via the Coulomb logarithm. In laboratory plasmas the effects of different ion species (impurities) has to be taken into account in a weighed average between the different charge numbers Z. ("$Z_{\rm eff}$")

A Appendix: Coulomb collisions

The equations of motion for two interacting particles are

$$m_1 \frac{d^2 \boldsymbol{x}_1}{dt^2} = \boldsymbol{F}_{12}, \qquad m_2 \frac{d^2 \boldsymbol{x}_2}{dt^2} = -\boldsymbol{F}_{12}, \tag{A.1}$$

where $\boldsymbol{F}_{12}$ is the force that particle 2 excerts on particle 1. In the present consideration $\boldsymbol{F}_{12}$ is a central force. One easily sees that if we define $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{w} = \dot{\boldsymbol{x}} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, the vector $\boldsymbol{x} \times \boldsymbol{w}$ is constant during the motion. Therefore, the collision process takes place in a plane perpendicular to $\boldsymbol{x} \times \boldsymbol{w}$ which includes the center of mass of the two particles. We multiply the two equations (A.1) by m_2 and m_1, respectively, and then subtract them to obtain

$$\mu \ddot{\boldsymbol{x}} = \boldsymbol{F}_{12} \tag{A.2}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. The collision can therefore be described in terms of the motion of a mass μ around a fixed center of force. Consider now the equations for the conservation of energy and angular

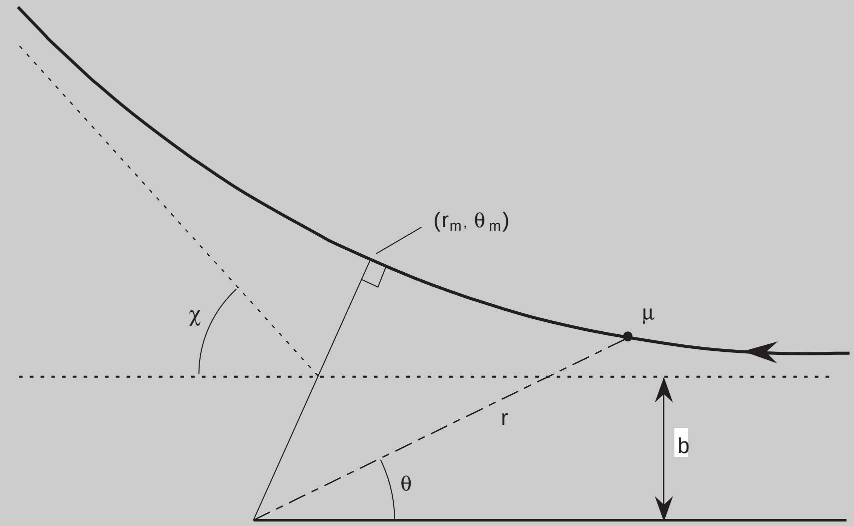


Figure 6: Motion with respect to a fixed central force.

momentum,

$$\tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \phi(r) = \tfrac{1}{2}\mu w_0^2, \qquad \mu r^2 \dot{\theta} = \mu b w_0. \tag{A.3}$$

Since we are not really interested in the particle positions as a function of time but only in the orbits, we eliminate time by writing $\dot{r}/\dot{\theta} = dr/d\theta$ and obtain

$$\frac{dr}{d\theta} = -\frac{r^2}{b}\sqrt{1 - \frac{b^2}{r^2} - 2\frac{\phi(r)}{\mu w_0^2}}, \tag{A.4}$$

where $w_0 = |\boldsymbol{v}_1 - \boldsymbol{v}_2|_0$ is the relative velocity of the particles long before the collision, b is the impact parameter, and $\phi(r)$ the interaction potential. The coordinates (r_m, θ_m) of the point of nearest approach, where $dr/d\theta = 0$, are given by the equations

$$1 - \frac{b^2}{r_m^2} - 2\frac{\phi(r_m)}{\mu w_0^2} = 0 \tag{A.5}$$

and

$$\theta_m = \int_{r_m}^{\infty} dr \, \frac{b/r^2}{\sqrt{1 - b^2/r^2 - 2\phi(r)/\mu w_0^2}}. \tag{A.6}$$

For the Coulomb interaction between electric charges q_1, q_2 with potential

$$\phi(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r} \tag{A.7}$$

equation (A.6) takes the form

$$\theta_m = \int_{r_m}^{\infty} dr \, \frac{b/r^2}{\sqrt{1 - 2b_0/r - b^2/r^2}}.$$

where

$$b_0 = \frac{q_1 q_2}{4\pi\epsilon_0 \mu w_0^2}$$

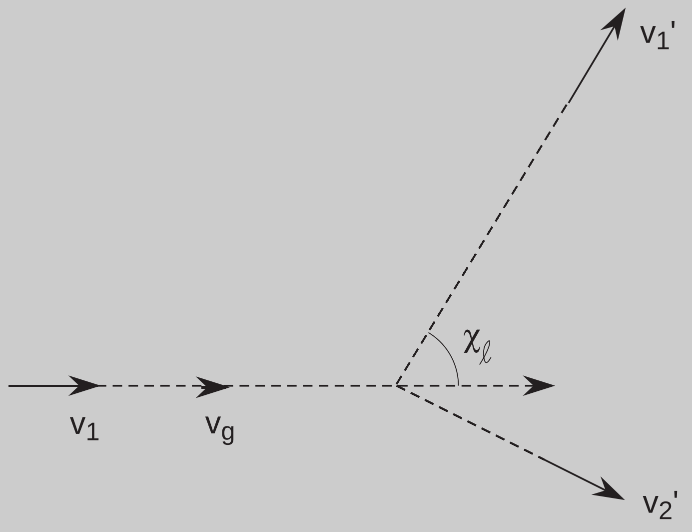
is the impact parameter for scattering over $90°$. The integral can be evaluated by making the substitution $c = (b_0 + b^2/r)/\sqrt{b_0^2 + b^2}$. One finds

$$\theta_m = \int_{b_0/\sqrt{b_0^2+b^2}}^{1} \frac{dc}{\sqrt{1-c^2}} = \arccos(c)\Big|_{b_0/\sqrt{b_0^2+b^2}}^{1} = \arctan(b/b_0).$$

Expressed in terms of the scattering angle $\chi = \pi - 2\theta_m$ one finds

$$\cot\frac{\chi}{2} = \frac{b}{b_0}, \tag{A.8}$$

Figure 7: The scattering angle χ_ℓ viewed in the frame in which particle 2 is initially at rest.



In the laboratory rest frame the particle velocities, expressed in terms of the velocity of the center of mass $\boldsymbol{v}_g$ and the relative velocity $\boldsymbol{w} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, are

$$\boldsymbol{v}_1 = \boldsymbol{v}_g + \frac{m_2}{m_1+m_2}\boldsymbol{w}, \qquad \boldsymbol{v}_2 = \boldsymbol{v}_g - \frac{m_1}{m_1+m_2}\boldsymbol{w} \tag{A.9}$$

If we now assume that particle 2 is initially at rest ($\boldsymbol{v}_{20} = 0$), the scattering angle χ_l in the laboratory frame is related to the scattering angle χ in the center of mass frame as

$$\tan\chi_l = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}. \tag{A.10}$$

The momentum change of particle 1 in the initial direction is therefore

$$\frac{\Delta m_1 v_1}{m_1 v_{10}} = -\frac{m_2}{m_1+m_2}(1-\cos\chi) = -\frac{2m_2}{m_1+m_2}\frac{1}{1+b^2/b_0^2}. \tag{A.11}$$

The relative transfer of kinetic energy from particle 1 to particle 2 is

$$\frac{\Delta m_2 v_2^2}{m_1 v_{10}^2} = \frac{4m_1m_2}{(m_1+m_2)^2}\frac{1}{1+b^2/b_0^2} \tag{A.12}$$

If the volume density of target particles 2 is n_2, the number of collisions per unit of time of test particle 1 with velocity v_{10} equals $n_2 v_{10}$. From (A.11) we find that the rate of momentum transfer per unit time is

$$\frac{1}{p}\frac{dp}{dt} = -2\frac{m_2 n_2}{m_1+m_2} v_{10}\, 2\pi \int_0^{\lambda_D} \frac{b\,db}{1+b^2/b_0^2} = -4\pi \frac{m_2 n_2}{m_1+m_2} w_0 b_0^2 \ln\Lambda, \tag{A.13}$$

where

$$\Lambda = \frac{\lambda_D}{b_0} = \frac{4\pi\epsilon_0^{3/2}\mu w_0^2 (kT)^{1/2}}{Z_1 Z_2 e^3 n^{1/2}} \gg 1. \tag{A.14}$$

The integral is cut off at $b = \lambda_d$ since, due to Debye shielding, larger impact parameters give a negligible contribution to the total scattering cross section. From (A.13) we find for the characteristic time for momentum transfer

$$\tau_c = \frac{1}{4\pi}\frac{m_1+m_2}{m_2 n_2}\frac{(4\pi\epsilon_0)^2\mu^2 w_0^3}{Z_1^2 Z_2^2 e^4\, \ln\Lambda}. \tag{A.15}$$

Likewise, from (A.12) we find the characteristic time for energy transfer,

$$\tau_E = \frac{m_1+m_2}{2m_1}\tau_c. \tag{A.16}$$

These collision times depend on the relative velocity w_0 of the interacting particles. The momentum and energy transfer per unit volume therefore depends on the velocity distributions of the interacting particle species.

If we consider a plasma in which the electron and ion populations are both thermal and substitute $\mu w_0^2 = 3kT_e$ for $e-i$ collisions and $\mu w_0^2 = 3kT_i$ for $i-i$ collisions, then we find the following characteristic times,

momentum transfer $e-i$:

$$\tau_e = 12\, 3^{1/2}\pi\, \frac{\epsilon_0^2 m_e^{1/2}(kT_e)^{3/2}}{n_i Z^2 e^4\, \ln\Lambda_e}, \tag{A.17}$$

momentum transfer $i-i$:

$$\tau_i = 12\, (\tfrac{3}{2})^{1/2}\pi\, \frac{\epsilon_0^2 m_i^{1/2}(kT_i)^{3/2}}{n_i Z^4 e^4\, \ln\Lambda_i}, \tag{A.18}$$

energy transfer $e-i$:

$$\tau_{ei} = \frac{m_i}{2m_e}\tau_e. \tag{A.19}$$

Apart from the precise numerical coefficients that can only be derived from kinetic theory, these results (A.17)–(A.19) agree with expressions (36)–(38).