Mid-term test Plasma Physics, 2 November 2009

The numerical calculations are in the accompanying Mathematica notebook (`midtermexam2009.nb`) and the .pdf-image thereof.

1 (40pt)

Consider charged particles in the dipole magnetic field around the Earth. In the equatorial plane, the field strength equals $B = m_E/R^3$ where the Earth's dipole moment is $m_E = 8.8 \times 10^{17}\,\mathrm{Tm}^3$ and R is the distance to the Earth's centre. The direction of $\boldsymbol{B}$ is north. Take the Earth's radius to be $R_E = 6500\,\mathrm{km}$ and the surface accelaration of gravity $9.8\,\mathrm{m/s}^2$.

(a) In which directions are the drift velocities for protons and electrons in the Earth's dipole field, due to the gravitational force?

The particles drift perpendicular to the field (north) and the force (downward).
The direction is opposite for electrons (westward) and protons (eastward).

(b) Give the gravitational drift velocity for electrons and for protons at a distance $3R_E$ from the Earth's centre, in the equatorial plane.

The gravitational drift equals $v_g = mg/qB$ (p. 20, lecture notes). Here $B = m_E/(3R_E)^3 = 1.187 \times 10^{-4}$T and the acceleration of gravity g is $(R/R_E)^2 = 9$ times smaller than on the Earth's surface. For electrons and protons:

$$v_{ge} = 5.216 \times 10^{-8}\text{m/s}, \qquad v_{gp} = 9.578 \times 10^{-5}\text{m/s}.$$

(c) At the same position, give the cyclotron frequencies for electrons and protons.

The cyclotron frequencies are given by $\Omega = qB/m$,

$$\Omega_{ce} = 2.087 \times 10^{7}\text{rad/s}, \qquad \Omega_{cp} = 1.137 \times 10^{4}\text{rad/s}.$$

(d) At the same position, give the gyro-radius of a proton with kinetic energy $E_\perp = 1\,\mathrm{eV}$ perpendicular to $\boldsymbol{B}$.

The perpendicular velocity is $v_\perp = \sqrt{2E_\perp/m_p}$. To get the result in SI units, use $1\,\mathrm{eV} = 1.6022 \times 10^{-19}$J. The gyro-radius is $\rho = m_p v_\perp/eB = \sqrt{2m_pE_\perp} = 1.22$m.

(e) For the proton in question 1(d), compute the ∇B drift velocity and clearly indicate its direction.

The ∇B-drift, given by the first term in Eq. (74) is

$$v_D = \frac{1}{2} m_p v_\perp^2 \frac{|\nabla B|}{eB^2} = \frac{E_\perp}{eB}\frac{|\nabla B|}{B} = \frac{3E_\perp}{eBR} = 1.296 \times 10^{-3}\text{m/s}.$$

2 (30 pt)

Consider hydrogen plasma in the solar photosphere with an electron density of $n_e = 10^{20}\mathrm{m}^{-3}$ and temperature $6000\,\mathrm{K}$.

(a) What is the ionization degree n_i/n_0 of hydrogen according to Saha's equation?
Eq. (2) of the lecture notes is a plug-in formula, taking the temperature T in Kelvin. The ionization-energy cooresponds to a temperature $E_i/k_B = 13.6$ eV $\times 1.16 \times 10^4$K/eV $= 1.58 \times 10^5$K. Hence,

$$\frac{n_i}{n_0} = 2.4 \times 10^{21} n_e^{-1} T^{3/2} e^{-E_i/k_B T} = 4.21 \times 10^{-5}.$$

(b) Compute the plasma frequency ω_p, the Debye length λ_D, and the number of particles N_D in the Debye sphere (sphere with radius λ_D).
The plasma frequency, the Debye length, and the plasma parameter are given by eqs. (6), (15), and (19), respectively. One finds

$$\omega_p = 5.64 \times 10^{11}\text{rad/s}, \qquad \lambda_D = 3.78 \times 10^{-7}\text{m}, \qquad N_D = 22.6.$$

Note that Eq. (15) for λ_D includes screening by electrons as well as ions. Formula (8), which includes only electron screening, gives a factor $\sqrt{2}$ higher value of λ_D and a factor $\sqrt{8}$ higher value of N_D. These results are accepted too.

(c) State the definition of an ideal plasma and indicate (with arguments) if the photospheric plasma is ideal.
A plasma is ideal if the kinetic energy of the particles are large compared to the interaction energy. The inequality $N_D \gg 1$ implies this. Since $N_D = 22.6 \gg 1$, the plasma is indeed ideal. Those who gave this inequality but doubted if 22.6 can be considered sufficiently large, also received full marks.

(d) Calculate the electron collision time τ_e and the mean free path λ_{mfp}.
Use the plug-in formulas (36) and (39) for τ_e and the Coulomb logarithm. These require the temperature in keV: $k_B T/e = 0.517$ eV $= 5.17 \times 10^{-4}$keV. The mean free path is $\lambda_{mfp} = \tau_e v_{th,e} = \tau_e \sqrt{k_B T/m_e}$. The results are

$$\tau_e = 1.68 \times 10^{-10}\text{s}, \qquad \lambda_{mfp} = 5.06 \times 10^{-5}\text{m}.$$

(e) Explain whether longitudinal plasma waves with angular frequency $\omega_1 = 10^9\,\mathrm{s}^{-1}$ propagate in this plasma. And with frequency $\omega_2 = 10^{12}\,\mathrm{s}^{-1}$? In case(s) where the wave propagates, compute the wave length $2\pi/k$. One sees from the dispersion relation (127) for longitudinal waves, $\omega^2 = \omega_{pe}^2(1 + \gamma k^2 \lambda_D^2)$, that waves only propagate for frequencies $\omega > \omega_p$, whih is the case for ω_2 but not for ω_1. Either expression in (127) can be used to calculate k.

$$k = \sqrt{(\omega^2 - \omega_{pe}^2)\frac{m_e}{\gamma k_B T}} = \frac{1}{\gamma^{1/2}\lambda_D}\sqrt{\frac{\omega^2}{\omega_{pe}^2} - 1}.$$

In Section 5.2 it is explained that $\gamma = 3$, which we take here, but $\gamma = 5/3$ is also accepted. The resulting wavelength is

$$\frac{2\pi}{k} = 3.975 \times 10^{-6}\text{m}.$$

Note that in Chapter 5, the Boltzmann constant k_B was systematically omitted, which means that the temperature was always understood to be the energy $k_B T$. No marks were subtracted for numerically different results in (2e) caused by omitting k_B.

3 (30 pt)

In a tokamak, a fusion plasma with central temperature $kT = 5 \times 10^3\,\mathrm{eV}$ and electron density $n_e = 10^{20}\mathrm{m}^{-3}$ is confined in a magnetic field of $5\,\mathrm{T}$. Assume that the electron collision time is $\tau_e = 7 \times 10^{-5}$s.

(a) Give the energy exchange time between electrons and ions.
Obtain the energy exchange time from Eq. (38) with $m_i = 2m_p$ for deuterons,

$$\tau_{ei} = \frac{m_i}{2m_e}\tau_e = 0.1285\mathrm{s}$$

(b) Give the electric resistivity in the plasma centre.
From Eq. (45) one obtains the resistivity

$$\eta = \frac{m_e}{n_e e^2 \tau_e} = 4.07 \times 10^{-9}\Omega\mathrm{m}.$$

(c) A loop voltage of $3\,\mathrm{V}$ is induced toroidally around the torus. Compute the induced current density in the plasma center (major radius $R = 6\,\mathrm{m}$). The loop voltage over a circular circumference of length $2\pi R$ implies an electric field $E = V/(2\pi R)$, which by Ohm's law yields an induced current density

$$J_{ind} = \frac{E}{\eta} = \frac{V}{2\pi R\eta} = 1.57 \times 10^7\mathrm{A/m}^2.$$

(d) Roughly estimate average density and temperature gradients from the fact that the plasma has a circular cross section with minor radius $r = 2\,\mathrm{m}$. Based on these estimates, compute the diamagnetic current density.
Basic estimates of the gradients are given by $|\nabla n| \approx n/r$ and $|\nabla T| \approx T/r$. The diamagnetic current can be found from Eqs. (96) or (97),

$$J_{dia} = \frac{|\nabla p|}{B} = \frac{2nk_BT}{rB} = 1.602 \times 10^4\mathrm{A/m}^2.$$

(e) Assume that the density and temperature are monotonic functions of the minor radius with a maximum in the plasma centre. Give qualitative sketches of the induced current density profile and the diamagnetic current density profile across a horizontal plasma cross section.
The resistivity is proportional to $\eta \propto T^{-3/2}$. Therefore the induced current density will be $J_{ind} \propto T^{3/2}$, peaked in the center of the plasma just like the temperature. The direction of the induced current is toroidal.
The diamagnetic current is proportional to ∇p and therefore zero at the plasma center, where p is maximal. Less obviously, ∇p also vanishes at the plasma edge. Therefore J_{dia} is maximal at intermediate radii where the pressure profile is steepest. It changes sign in the plasma center.
(More precisely: the direction of $\boldsymbol{J}$ is almost poloidal since it is perpendicular to $\boldsymbol{B}$, which is almost toroidal, and ∇p, which is radial.)