

## Solution exercise “Magnetic islands”

An axisymmetric magnetic field can always be written as

$$\mathbf{B} = \nabla\phi \times \nabla\psi + F(\psi)\nabla\phi,$$

where  $\phi$  is the toroidal angle and  $\nabla\phi \cdot \nabla\psi = 0$ .

**a. Show that  $\nabla \cdot \mathbf{B} = 0$  and the field lines lie in the surfaces given by  $\psi = \text{constant}$ .**

We have  $\nabla \cdot \mathbf{B} = 0$  because

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = (\nabla \times \nabla\phi) \cdot \nabla\psi - (\nabla \times \nabla\psi) \cdot \nabla\phi = 0, \quad \nabla \cdot (F\nabla\phi) = \frac{dF}{d\psi} \nabla\psi \cdot \nabla\phi = 0.$$

Field lines lie in constant- $\psi$  surfaces because  $\mathbf{B} \cdot \nabla\psi = 0$ .

**Show that on any given magnetic surface the toroidal field component is inversely proportional to the distance to the symmetry axis.**

On a constant- $\psi$  surface,  $B_{tor} = F(\psi)|\nabla\phi|$  is proportional to  $|\nabla\phi| = 1/R$ .

**b. We now build a simplified, “straight” model of the magnetic surfaces that imitate a torus with minor radius  $r$  and major radius  $R$ . To this end we introduce cartesian coordinates  $(x, y, z)$ :  $x$  is the radial coordinate,  $y$  is the poloidal coordinate (periodic for  $y \rightarrow y + 2\pi r$ ), and  $z$  is the toroidal coordinate (periodic for  $z \rightarrow z + 2\pi R$ ). The axisymmetric field becomes**

$$\mathbf{B} = (\hat{\mathbf{z}} \times \nabla\psi + F(x)\hat{\mathbf{z}})/R$$

**This simplifies to  $\mathbf{B} = (F(x)\hat{\mathbf{z}} + (d\psi/dx)\hat{\mathbf{y}})/R$  because  $\psi = \psi(x)$ . Show that the safety factor is  $q(x) = rF/(R d\psi/dx)$ . Compute the current density  $\mathbf{J}$ .**

Following a field line, the ratio between “toroidal revolutions” and “poloidal revolutions” is

$$q = \frac{2\pi r}{2\pi R} \frac{dz}{dy} = \frac{r}{R} \frac{B_z}{B_y} = \frac{rF}{R(d\psi/dx)}.$$

**c. Consider the special case  $F(x) = B_0 R$  and  $\psi(x) = \frac{1}{2} J_0 \mu_0 R x^2$ . Show that in this case the current density is constant.**

The current density is

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} = \frac{1}{\mu_0} \nabla \times (B_0 \hat{\mathbf{z}} + J_0 \mu_0 x \hat{\mathbf{y}}) = J_0 \hat{\mathbf{x}} \times \hat{\mathbf{y}} = J_0 \hat{\mathbf{z}}.$$

**Compute the safety factor profile  $q(x)$  and the magnetic shear  $s \equiv (dq/dx)x/q$ .**

The safety factor is  $q(x) = rB_0/(J_0 \mu_0 R x)$  so that  $s(x) = -1$ .

**d. Take once more  $F(x) = B_0 R$  so that the magnetic field has the form**

$$\mathbf{B} = \hat{\mathbf{z}} \times \nabla\psi/R + B_0 \hat{\mathbf{z}}.$$

**However, now introduce a helical symmetric perturbation with poloidal wave number  $m$  and toroidal wave number  $n$  by introducing  $y, z$  dependencies in  $\psi$ :**

$$\psi = \frac{1}{2} J_0 \mu_0 R x^2 + \psi_1 \cos\left(\frac{m}{r} y - \frac{n}{R} z\right).$$

**Show that still  $\nabla \cdot \mathbf{B} = 0$ , but that  $\psi = \text{constant}$  no longer determines a magnetic surface.**

$$\nabla \cdot \mathbf{B} = (\nabla \times \nabla\psi) \cdot \hat{\mathbf{z}}/R = 0.$$

We conclude that field lines do not lie in  $\psi=\text{constant}$  surfaces because:

$$\mathbf{B} \cdot \nabla\psi = B_0 \hat{\mathbf{z}} \cdot \nabla\psi = B_0 \frac{\partial\psi}{\partial z} = \frac{B_0 \psi_1 n}{R} \sin\left(\frac{m}{r}y - \frac{n}{R}z\right) \neq 0.$$

**e. It turns out, that magnetic surfaces are given by  $\psi^*=\text{constant}$  with**

$$\psi^* = \frac{1}{2} J_0 \mu_0 R (x - x_r)^2 + \psi_1 \cos\left(\frac{m}{r}y - \frac{n}{R}z\right).$$

**for a suitable value of  $x_r$ . Determine this so-called resonance radius  $x_r$  and show that  $\psi^*=\text{constant}$  indeed gives magnetic surfaces. Determine  $q(x_r)$ .**

In order for  $\psi^*$  to determine magnetic surfaces, we need to have  $\mathbf{B} \cdot \nabla\psi^* = 0$ . For arbitrary  $x_r$  we have

$$\begin{aligned} \mathbf{B} \cdot \nabla\psi^* &= \frac{1}{R} (\hat{\mathbf{z}} \times \nabla\psi) \cdot \nabla(\psi^* - \psi) + B_0 \hat{\mathbf{z}} \cdot \nabla\psi = -\frac{1}{R} (\hat{\mathbf{z}} \times \nabla\psi) \cdot J_0 \mu_0 R x_r \hat{\mathbf{x}} + B_0 \hat{\mathbf{z}} \cdot \nabla\psi \\ &= x_r J_0 \mu_0 \hat{\mathbf{y}} \cdot \nabla\psi + B_0 \hat{\mathbf{z}} \cdot \nabla\psi = -\psi_1 \left( J_0 \mu_0 x_r \frac{m}{r} + B_0 \frac{n}{R} \right), \end{aligned}$$

which vanishes if  $x_r = B_0 n r / (J_0 \mu_0 m R)$ . At that position  $q(x_r) = m/n$ .

**f. Sketch lines of constant  $\psi^*$  in the  $(x, y)$ -plane in the neighbourhood of  $x = x_r$ . Assume for convenience that  $J_0$  and  $\psi_1$  are both positive. What is special about the line(s)  $\psi^* = \psi_1$ ?**

Around  $x = x_r$ ,  $\psi^*$  increases in the positive and negative  $x$ -directions. On the line  $x = x_r$ ,  $\psi^*$  oscillates in the  $y$ -direction between minima  $\psi^* = -\psi_1$  and maxima  $\psi^* = \psi_1$ . So the points  $x = x_r, y = 2\pi k r / m$  are minima in  $x$  and maxima in  $y$ . They are therefore saddle points, where two contours  $\psi^* = \psi_1$  intersect. Hence the name ‘‘X-point’’. The equation for these contours is

$$\begin{aligned} \psi_1 &= \frac{J_0 \mu_0 R}{2} (x - x_r)^2 + \psi_1 \cos(my/r) \implies \\ x(y) &= x_r \pm \sqrt{\frac{\psi_1}{J_0 \mu_0 R}} \sqrt{\frac{1 - \cos(my/r)}{2}} = x_r \pm \sqrt{\frac{\psi_1}{J_0 \mu_0 R}} \sin(my/2r). \end{aligned}$$

**What is special about the point(s)  $\psi^* = -\psi_1$ ?**

These points are local minima in  $(x, y)$ , and therefore surrounded by closed contours  $\psi^*=\text{constant}$ . Hence the name ‘‘O-point’’.

**g. What is the maximal distance in the  $x$ -direction between the  $\psi^* = \psi_1$  lines?**

From the expression for  $x(y)$  in answer (f) one sees that the magnetic island width is  $2(\psi_1 / J_0 \mu_0 R)^{1/2}$ .

**h. Give the current density in the  $z$ -direction including perturbation.**

$$J_z = \frac{1}{\mu_0} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) = \frac{1}{\mu_0 R} \left( \frac{\partial^2 \psi^*}{\partial x^2} + \frac{\partial^2 \psi^*}{\partial y^2} \right) = J_0 - \frac{\psi_1 m^2}{\mu_0 R r^2} \cos(my/r).$$

**For which value of  $y$  is the current density maximal? For which value of  $y$  is it minimal? (CORRECTION: original question asked for  $z$ )**

The current is minimal for  $y = 2\pi k r / m$ , i.e., in the X-points. The current is maximal for  $y = \pi(2k + 1)r / m$ , i.e., in the O-points.