

# Plasma parameters

Mathematica notebook, accompanying Chapters 1 and 2 of the Plasma Physics Lecture Notes.

periods 1 and 2, 2009/2010, Utrecht University.

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Evaluate all parts of this notebook in the given order.  
Formulas are given for quantitative calculations of a number of plasma parameters, for given densities and temperatures.  
Units are SI, except the temperature, where this is indicated.

## ■ Formatting macros

## ■ Physical constants

```
me = electron mass [kg]
mp = proton mass [kg]
mD = Deuteron mass [kg]
ee = electron charge [C] = 1 eV / J
c = speed of light [m/s]
μ0 = magnetic permittivity [H/m]
k = Boltzmann constant [J/K]
h = Planck's constant [J s]
ei = ionization energy of H [eV]
eF = Fermi energy [eV]
μD = mass ratio Deuteron/proton
ε0 = dielectric constant [F/m]
ev = temperature of 1 eV [K/eV] (11600.)
```

```
me = 9.1094 × 10-31;
mp = 1.6726 × 10-27;
mD = 3.3436 × 10-27;
ee = 1.6022 × 10-19;
c = 2.99793 × 108;
μ0 = 4 π 10-7;
k = 1.3807 × 10-23;
h = 6.6261 × 10-34;
ei = ee3 me / (8 ε02 h2);
eF = (h / 2 / π)2 (3 π2 n)2/3 / (2 me ee);
μD = mD / mp;
ε0 = μ0-1 c-2;
ev = ee / k;
{μD, ε0, ev}

{1.99904, 8.85416 × 10-12, 11604.3}
```

## ■ Saha equation

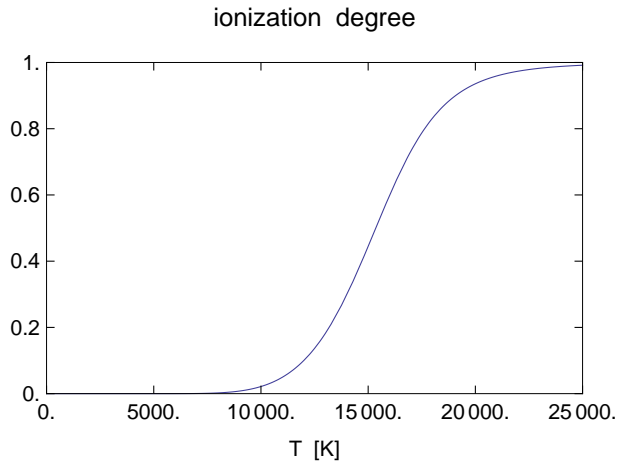
Saha equation; Figure 1 in lecture notes.

```
ni = ion density [particles/m3]
t = temperature [K]
kt = temperature [eV]
cS = the numerical factor in Eq. (2) of the lecture notes (printed below)
saha = ni ne / n0; here: ne=ni. It's the right-hand side of Eq. (2).
id = ni / (n0 + ni), ionization degree
p = ee(n0+ne+ni)kT, total pressure [Pa]
```

```

p = 105;
cS = (  $\frac{2 \pi m e e e^{3/2}}{h^2 e v}$  ) ;
{cS, 3  $\pi^{1/2} (eF e v)^{-3/2} n / 8$ } (* print *)
saha = cS t3/2 Exp[-ei ev / t];
id = 1 / Sqrt[1 + pev / (ee saha t)];
sahaplot = TidyPlot[id, {t, 0, 25 000}, PlotRange → {{0, 25 000}, {0, 1}},
  FrameLabel → {"T [K]", ""}, PlotLabel → "ionization degree"]
Export["saha6.eps", sahaplot, "EPS"];
{2.41479 × 1021, 2.41479 × 1021}

```



Bittencourt's fig. 10 on page 193

```

(* nt = n0 + ni = 1016, 1019, 1022, 1025 *)
sahab = cS (kt ev)3/2 Exp[-ei / kt];
idb = 1 / (Sqrt[0.25 + nt / sahab] + 0.5);
lid = N[{idb /. nt → 1016, idb /. nt → 1019, idb /. nt → 1022, idb /. nt → 1025};
TidyPlot[lid, {kt, 0, 2.2}, PlotRange → {{0, 2.2}, {0, 1}}, Frame → True,
  FrameLabel → {"T [eV]", ""}, PlotLabel → "ionization fraction α"]

```

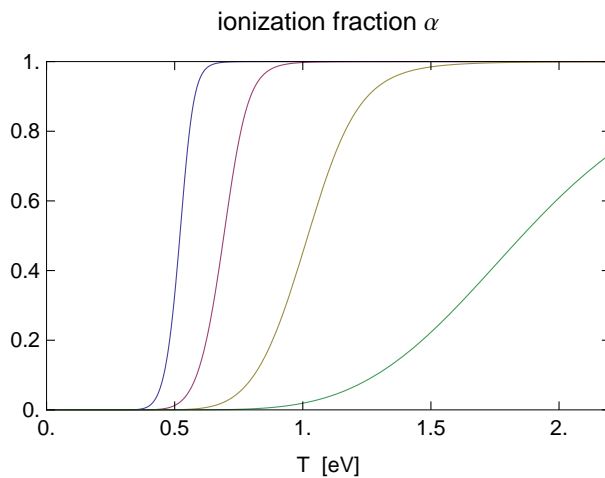
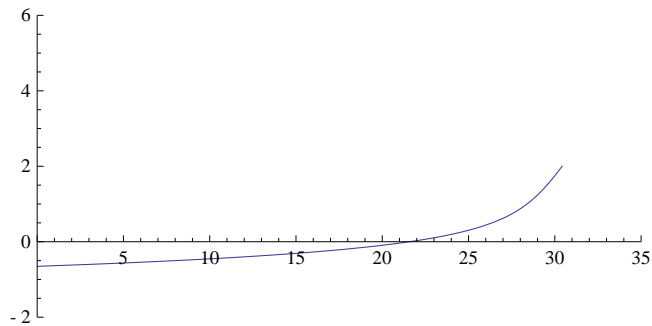


Figure 4 in the lecture notes contains the 50% ionization line  $n_i = n_0$ .

```
ParametricPlot[{Log[10, sahab], Log[10, kt]},
  {kt, 0.1, 100}, PlotRange -> {{0, 35}, {-2, 6}}, AspectRatio -> 0.5]
```



### ■ Exercise 1a

Compute the ionization degree  $n_i / n_0$  in the following 3 cases (densities in  $\text{m}^{-3}$ , temperatures in K)

(a) Room temperature,  $n_i \ll n_0$ .

```
ionization = (saha / n0)^{1/2} /. {t -> 300, n0 -> 3 * 10^{25}}
3.35337 * 10^{-115}
```

(b) Fusion reactor,  $n_i \gg n_0$ .

```
ionization = saha / ni /. {t -> 10^8, ni -> 10^{20}}
2.41098 * 10^{13}
```

(c) Core of the sun, start with neutral density

```
ionsun = (saha / n0)^{1/2} /. {t -> 1.6 * 10^7, n0 -> 10^{32}}
1.23705
```

(c') Core of the sun. We see that  $n_0$  and  $n_i$  have comparable magnitudes.

That's why we get a different answer when we start with the ion density.

Technical note: It may be surprising that the calculated ionization degree in the core of the sun is so low. In reality the ionization degree is higher.

The reason is, that the density in the core of the sun is so high that each neutral particle is having short-range interactions with many nearest neighbour particles simultaneously, whereas the Saha equation is based on the equilibrium between simple ionization and recombination reactions that involve two-particle interactions only.

```
ionization = saha / ni /. {t -> 1.6 * 10^7, ni -> 10^{32}}
1.53029
```

### ■ Collision times

OLD: Formulas (36) - (41) in the lecture notes are taken from Wesson's book "Tokamaks",

page 41 tables 2.9.1 - 2.9.3 and in the appendix.

The numerical plug-in formulas take:

densities in [ $10^{20} \text{ m}^{-3}$ ],

temperatures in [keV].

We print the numerical factors in (36), (37), and (38)

```

 $\lambda_{ei} = 15.2 - 1/2 \text{Log}[n] + \text{Log}[te];$ 
 $\lambda_{ee} = 14.9 - 1/2 \text{Log}[n] + \text{Log}[te];$ 
 $\lambda_{ii} = 17.3 - 1/2 \text{Log}[n] + 3/2 \text{Log}[ti];$ 
 $ce = 6 \times 2^{1/2} \pi^{3/2} \epsilon_0^2 me^{1/2} ee^{-5/2};$ 
 $cp = 2^{1/2} (mp/me)^{1/2} ce;$ 
 $cpe = (mp/2/me) ce;$ 
sci[109/2 {ce, cp, cpe}]

{1.08802 × 1016, 6.59333 × 1017, 9.98874 × 1018}

```

NEW: Now we adapt these formulas so that they take:

$n$  = density in [ $m^{-3}$ ],

$te$  = electron temperature in [eV].

$ti$  = ion temperature in [eV].

Let's print the numerical factors as they would appear in formulas (36), (37), and (38)

to make them suitable for temperature in [eV].

```

 $\lambda_{ei} = 15.2 - 1/2 \text{Log}[10^{-20} n] + \text{Log}[10^{-3} te];$ 
 $\lambda_{ee} = 14.9 - 1/2 \text{Log}[10^{-20} n] + \text{Log}[10^{-3} te];$ 
 $\lambda_{ii} = 17.3 - 1/2 \text{Log}[10^{-20} n] + 3/2 \text{Log}[10^{-3} ti];$ 
 $te = ce te^{3/2} / n / \lambda_{ee};$ 
 $ti = cp ti^{3/2} / n / \lambda_{ii};$ 
 $tei = (mp/2/me) te;$ 
sci[{ce, cp, cpe}]

{3.44063 × 1011, 2.08499 × 1013, 3.15872 × 1014}

```

Next we print out table 2 on page 14

```

sci[TableForm[te /. n → {1019, 1020} /. te → {102, 103, 104}]]

2.50251 × 10-6 0.0000677842 0.00187461
2.73122 × 10-7 7.30217 × 10-6 0.000200007

sci[TableForm[ $\mu D tei$  /. n → {1019, 1020} /. te → {102, 103, 104}]]

0.00459273 0.124401 3.44037
0.000501247 0.0134013 0.367062

sci[TableForm[ $\mu D^{1/2} ti$  /. n → {1019, 1020} /. ti → {102, 103, 104}]]

0.000196562 0.00050523 0.134576
0.0000212906 0.000538852 0.0142042

```

## ■ Plasma parameters

Plasma parameters, taken from the NRL plasma formulary, and listed in Table 1 of the lecture notes.

te = electron temperature (eV)  
ti = ion temperature (eV)  
n = density (m<sup>-3</sup>)  
r0 = average nearest neighbour distance [m]  
ωp = plasma frequency [rad/sec]  
de = c/ωp inertial skin depth [m]  
vte = electron thermal velocity [m/s]  
vti = ion thermal velocity [m/s]  
λL = Landau length [m]  
λD = Debye length [m]  
nD = plasma parameter  
λe = electron mean free path  
λi = ion mean free path

```
r0 = n-1/3;  
ωp = ee Sqrt[n / me / ε0];  
de = c / ωp;  
vte = (ee te / me)1/2 ;  
vti = (ee ti / mp)1/2 ;  
λL = ee2 / (6 π ε0 ee te);  
λD = (ε0 te / n / ee)1/2 ;  
nD = 4 π n λD3 / 3;  
λe = vte te;  
λi = vti ti;
```

Parameters for Table 2

```

plist[parms_] := {parms[[1]], 10-6 n, te, wp, λD, nD, 1 / te} /.
  {n -> parms[[2]], te -> parms[[3]], ti -> parms[[3]]};
plasmadata = {
  {"interstellargas", 106, 1},
  {"gaseousnebula", 109, 1},
  {"solarcorona", 1015, 102},
  {"diffusehot", 1018, 102},
  {"solaratmosphere", 1020, 1},
  {"gasdischarge", 1020, 10},
  {"hotplasma", 1020, 102},
  {"thermonuclear", 1021, 104},
  {"densehotplasma", 1024, 102},
  {"laserplasma", 1026, 102}};
sci[TableForm[Map[plist, plasmadata]]]

```

interstellargas	1.	1.	56415.5	7.43387	1.72082 × 10 <sup>9</sup>	0.0000700753
gaseousnebula	1000.	1.	1.78401 × 10 <sup>6</sup>	0.23508	5.4417 × 10 <sup>7</sup>	0.0600368
solarcorona	1. × 10 <sup>9</sup>	100.	1.78401 × 10 <sup>9</sup>	0.0023508	5.4417 × 10 <sup>7</sup>	53.3445
diffusehot	1. × 10 <sup>12</sup>	100.	5.64155 × 10 <sup>10</sup>	0.0000743387	1.72082 × 10 <sup>6</sup>	43306.
solaratmosphere	1. × 10 <sup>14</sup>	1.	5.64155 × 10 <sup>11</sup>	7.43387 × 10 <sup>-7</sup>	172.082	2.3229 × 10 <sup>9</sup>
gasdischarge	1. × 10 <sup>14</sup>	10.	5.64155 × 10 <sup>11</sup>	2.3508 × 10 <sup>-6</sup>	5441.7	9.46195 × 10 <sup>7</sup>
hotplasma	1. × 10 <sup>14</sup>	100.	5.64155 × 10 <sup>11</sup>	7.43387 × 10 <sup>-6</sup>	172082.	3.66136 × 10 <sup>6</sup>
thermonuclear	1. × 10 <sup>15</sup>	10000.	1.78401 × 10 <sup>12</sup>	0.000023508	5.4417 × 10 <sup>7</sup>	46652.1
densehotplasma	1. × 10 <sup>18</sup>	100.	5.64155 × 10 <sup>13</sup>	7.43387 × 10 <sup>-8</sup>	1720.82	2.3229 × 10 <sup>10</sup>
laserplasma	1. × 10 <sup>20</sup>	100.	5.64155 × 10 <sup>14</sup>	7.43387 × 10 <sup>-9</sup>	172.082	1.65367 × 10 <sup>12</sup>

### ■ Exercise 1b

Compute the electron-electron Coulomb logarithm  $\lambda_{ee}$ , the electron collision time  $\tau_e$ , and the plasma frequency  $\omega_p$  for:  
 (b) a fusion reactor

```

{λee, sci[τe], sci[ωp]} /. te -> k t / ee /. {n -> 1020, t -> 108}
{17.0538, 0.000161395, 5.64155 × 1011}

```

(c) Core of the sun, for simplicity take  $n_e = n_0$

```

{λee, sci[τe], sci[ωp]} /. te -> k t / ee /. {n -> 1032 ionsun, t -> 1.6 × 107}
{1.29934, 1.09593 × 10-16, 6.27468 × 1017}

```

### ■ Exercise 1e

Saha:  $n_i/n_0 = (3/8) \text{Sqrt}[\pi] (kT / eF)^{(3/2)} \text{Exp}[-e_i / kT]$

Show that  $n_D = (1/4) \text{Sqrt}[\pi/6] (kT / \text{Sqrt}[e_i eF])^{(3/2)}$

Then  $n_D = 1$  and  $kT = eF \Rightarrow kT = (8/\text{Sqrt}[6\pi])^{(4/3)} e_i$   
 and Saha yields

```

ni/n0 = (3/8) Sqrt[π] Exp[-(Sqrt[6π]/8)(4/3)]

```

```
FullSimplify[nD (Sqrt[eF ei ] / te)3/2]
```

$$0.1809 n \left( \frac{n^{1/3}}{te} \right)^{3/2} \left( \frac{te}{n} \right)^{3/2}$$

```
Sqrt[π / 6] / 4.0
```

```
0.1809
```

```
N[(3 / 8) Sqrt[π] Exp[-(Sqrt[6 π] / 8)4/3]]
```

```
0.42693
```

```
solve = Solve[{nD == 1, te == eF}, {n, te}] [[1]]
```

```
{n → 6.96515 × 1030, te → 132.99}
```

```
eVsaha = saha /. t → te ev;  
(eVsaha / n) /. solve
```

```
0.600031
```