



Plasma Equilibrium in Tokamaks

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Outline:

- Ideal MHD equilibrium
- The equilibrium problem
- Axisymmetry: Grad-Shafranov equation
- Safety factor
- Flux coordinates
- Shafranov shift

The tokamak magnetic field

It is important to know the tokamak magnetic field accurately

- The field determines the particle orbits
- The field determines energy and particle confinement

The magnetic field in tokamaks is determined by

1. Currents in the coil system
2. Conductors near the plasma
3. The plasma current distribution
4. The plasma pressure distribution

- Challenges:

- The plasma current distribution is hard to measure accurately
- The plasma pressure depends on the magnetic field ($\mathbf{B} \cdot \nabla p \approx 0$)

MHD equilibrium

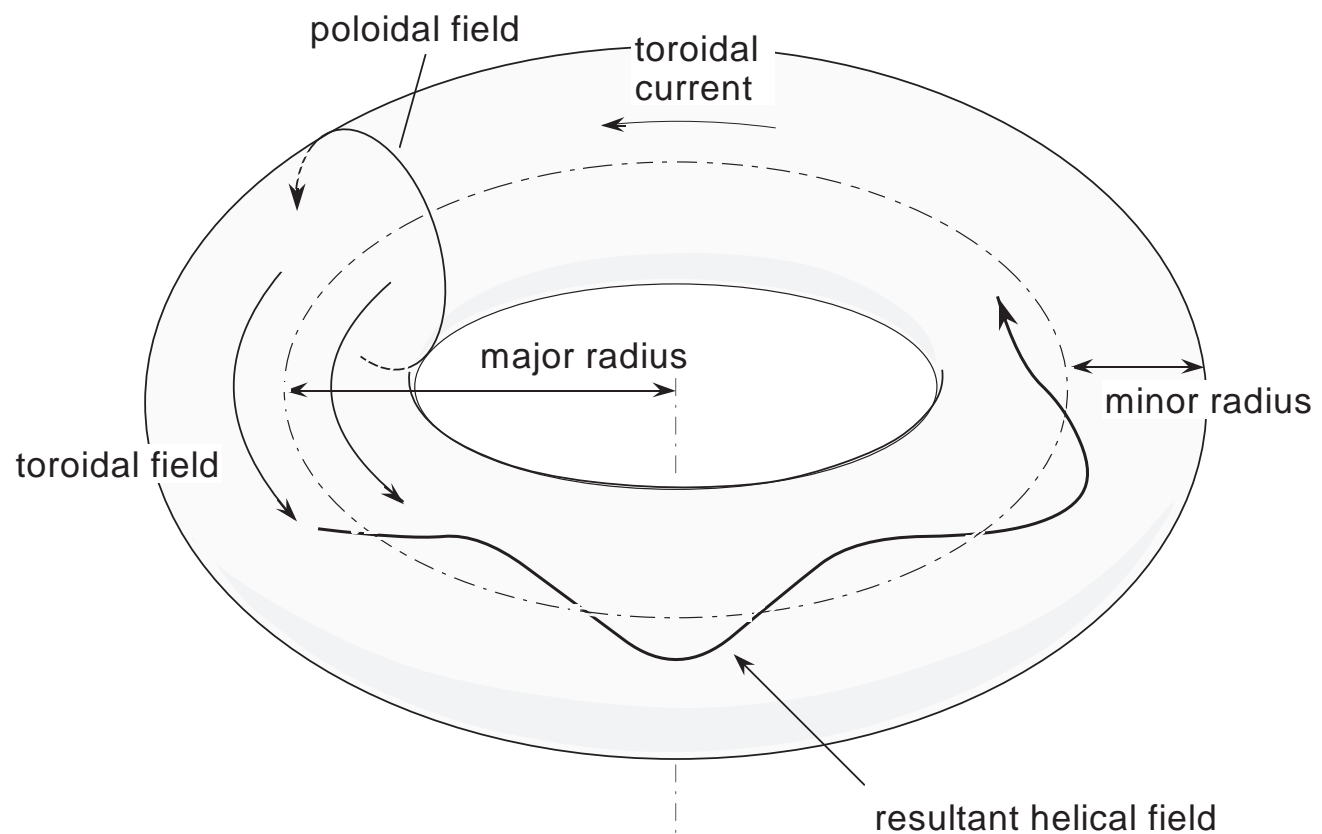
Ideal MHD equilibria are determined by the balance between plasma pressure and magnetic force,

$$\nabla p = \mathbf{j} \times \mathbf{B} \quad \text{with} \quad \mathbf{j} = \nabla \times \mathbf{B}.$$

Hopf's theorem: In a bounded region where $\nabla p \neq 0$, the force balance (with the conditions $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{j} = 0$) implies:

- The surfaces of constant p are tori (simply nested or braided)
- The field lines of \mathbf{B} and \mathbf{j} lie in these surfaces.
- The field lines either:
 - close after a finite number of toroidal revolutions
 - cover the surface ergodically.

Axisymmetry



- The tokamak fields are approximately axisymmetric.
- We shall neglect some sources of errors:
 - The toroidal field is generated by a discrete set of coils
 - “Tokamak equilibrium” usually slowly evolves (plasma heating, cooling, current diffusion)
 - “Tokamak equilibrium” usually describes a turbulent plasma, including current and magnetic field fluctuations.
- MHD equilibrium is a fair description of the intermediate timescales.
- Fluctuations are relatively unimportant for toroidal averages.

Axisymmetric equilibrium

In order to describe axisymmetric equilibria, we use the following coordinate system:

Since $\partial/\partial\phi = 0$, Gauss' law is

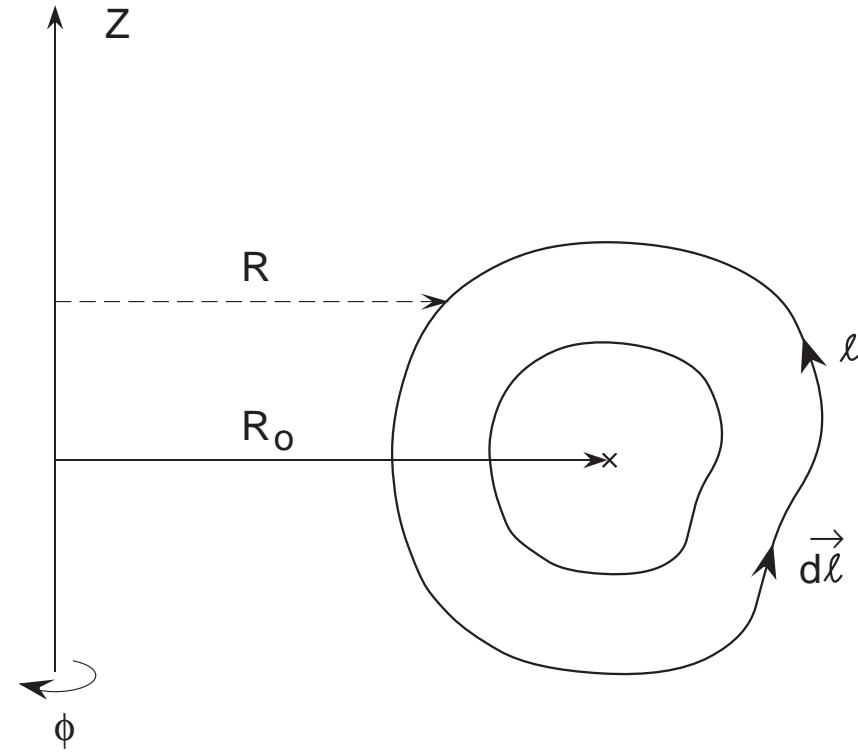
$$0 = \nabla \cdot \mathbf{B} = \frac{1}{R} \frac{\partial}{\partial R} (R B_R) + \frac{\partial B_Z}{\partial Z},$$

$\Rightarrow \mathbf{B}$ in terms of a stream function ψ :

$$\mathbf{B} = \mathbf{B}_p + B_t \mathbf{e}_\phi; \quad \mathbf{B}_p = \nabla \psi \times \nabla \phi,$$

($\mathbf{e}_\phi = R \nabla \phi$ is the toroidal unit vector)

- \mathbf{B} lies in surfaces (tori) of constant ψ : $\mathbf{B} \cdot \nabla \psi = 0$
- $\mathbf{B} \cdot \nabla p = 0 \Rightarrow p = p(\psi)$



The Grad-Shafranov equation

Ampère's law gives the current density,

$$\mathbf{j} = \nabla \times \mathbf{B} = -\Delta^* \psi \nabla \phi + \nabla(RB_t) \times \nabla \phi,$$

where the elliptic, Laplacian-like Grad-Shafranov operator Δ^* is defined by

$$\Delta^* \psi = R^2 \nabla \cdot \left(\frac{\nabla \psi}{R^2} \right) = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}.$$

From $\mathbf{j} \cdot \nabla p = 0$ one sees that also RB_t is a surface quantity,

$$RB_t = F(\psi).$$

The force balance becomes a p.d.e. for $\psi(R, Z)$:

$$\Delta^* \psi = -R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi} \quad (= Rj_t).$$

The Grad-Shafranov equation (II)

$$\Delta^* \psi = -R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi} .$$

This p.d.e. can be solved for $\psi(R, Z)$ after specifying:

- the pressure profile $p(\psi)$
- the profile $F(\psi)$
- the value of ψ everywhere on a closed contour
(e.g. by specifying the shape of one flux contour)

Challenges:

- One can measure some values $p(R, Z)$ but not $p(\psi)$
- $F(\psi)$ is even more difficult to measure.
- $j_t = -Rp' - FF'/R$ is **not** a flux function

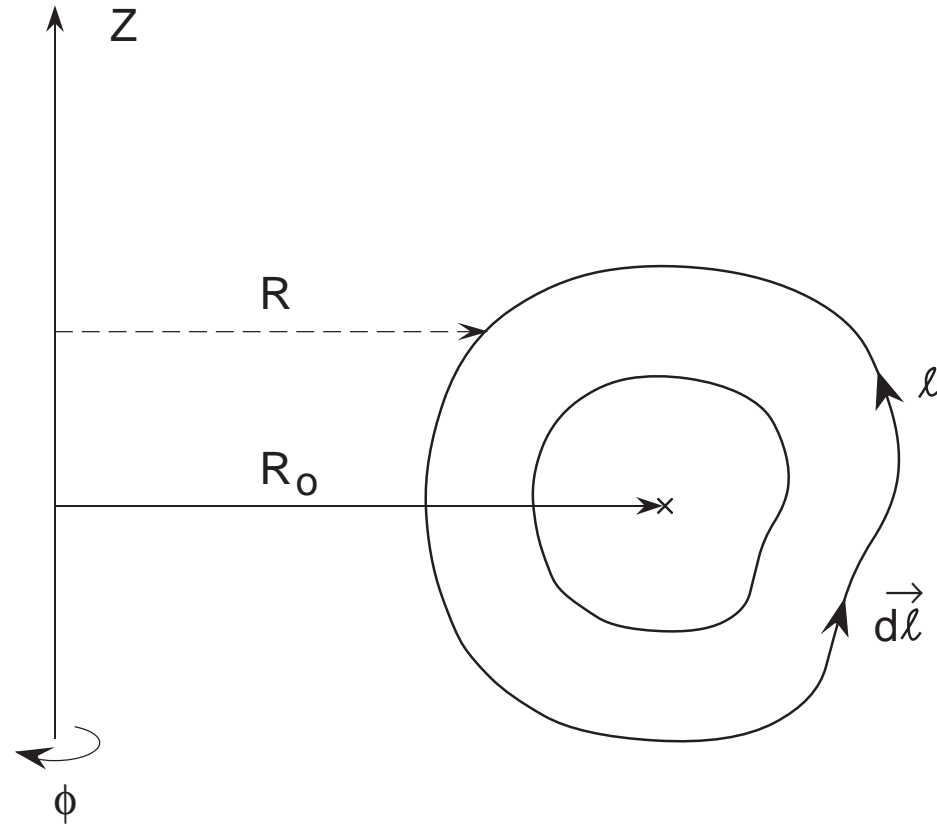
The equilibrium problem

The method of determining the magnetic equilibrium depends on:

- **The available measurements and other data**
 - The externally applied fields
 - The induced plasma current
 - the poloidal field outside and inside the plasma
 - the pressure distribution
 - quantities/phenomena that are tied to (specific) magnetic surfaces.
- **the purpose of the equilibrium reconstruction**
 - Fast reconstruction during a discharge (control of plasma position, RF-heating, etc.)
 - Interpretation of some plasma measurements, e.g. line-integrated data.
 - Simulation of plasma turbulence, heating, and transport
 - Stability analysis and calculation of instabilities

The safety factor q

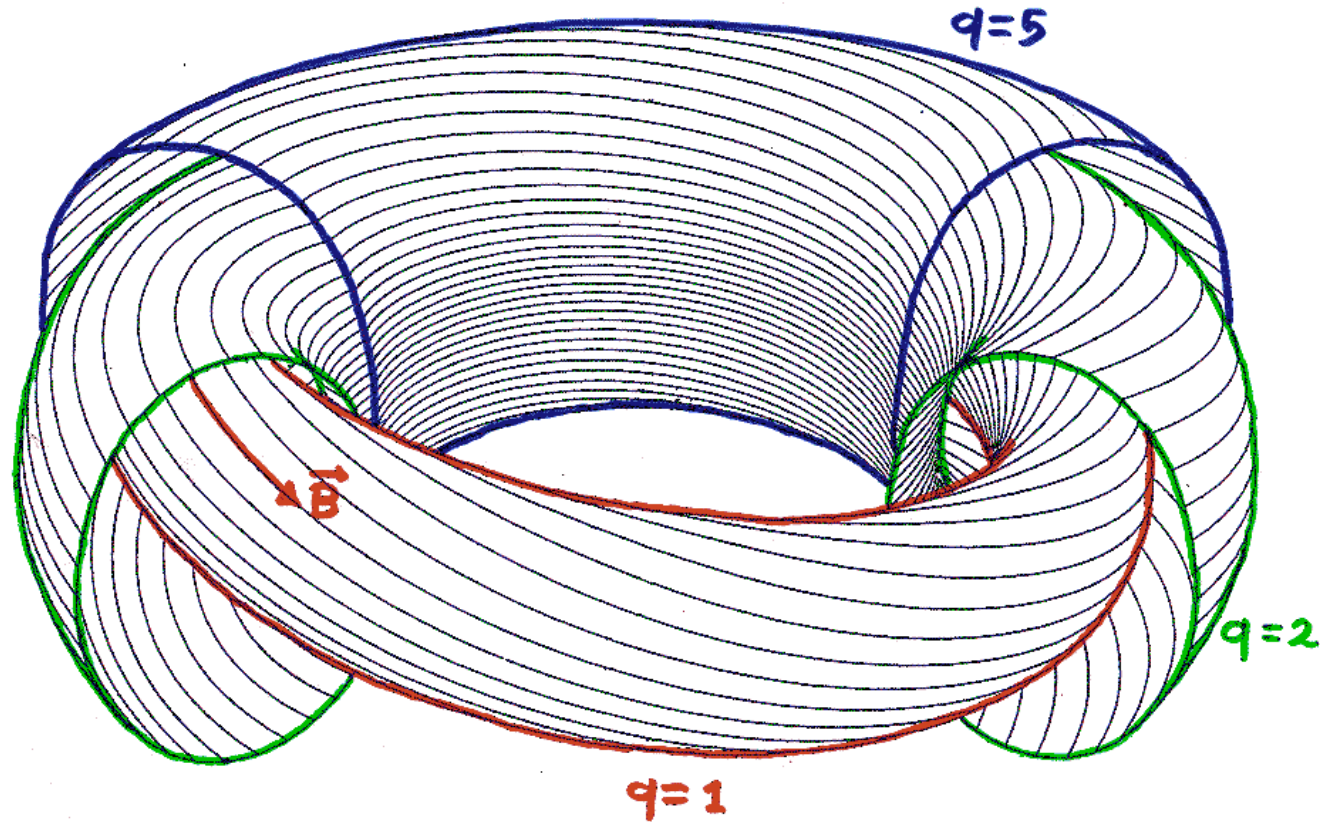
An important flux function is the ratio between toroidal and poloidal windings of the field lines on a surface,



$$q(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{d\theta} d\theta = \frac{1}{2\pi} \oint \frac{B_t}{R} \frac{d\ell}{B_p} = \frac{F}{2\pi} \oint \frac{d\ell}{R|\nabla\psi|}$$

The safety factor q

Some typical values of q :



The plasma current distribution determines the magnetic shear.

Flux surface averages

The flux surface average $\langle X \rangle$ of a given quantity X is the volume average between two neighbouring flux surfaces:

$$\langle X \rangle = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \int_{\psi}^{\psi + \Delta \psi} X dV = \frac{\oint X \tilde{f}(d\ell / B_p)}{\oint \tilde{f}(d\ell / B_p)}$$

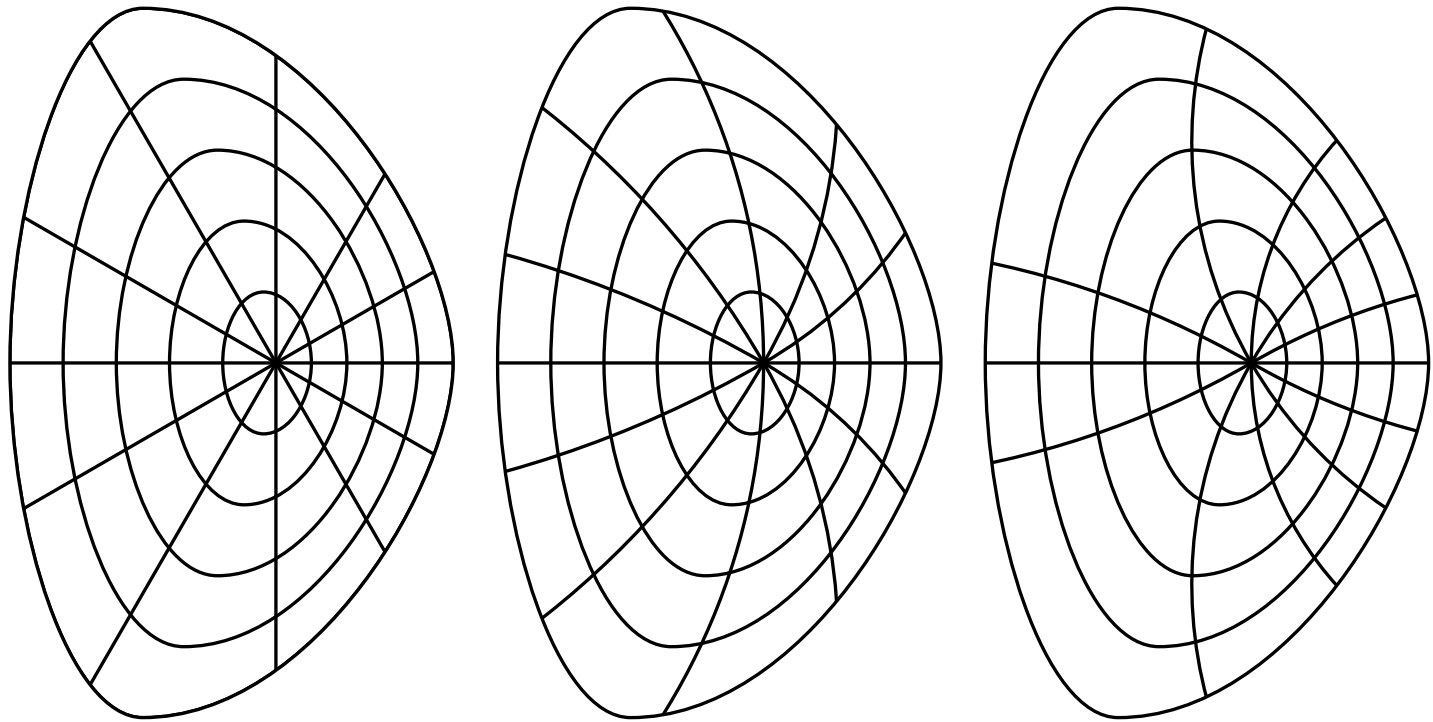
Also q can be written as a surface average,

$$q = \frac{F}{4\pi^2} \left\langle \frac{1}{R^2} \right\rangle \frac{dV}{d\psi}.$$

Flux coordinates consist of:

1. A radial coordinate $r(\psi)$; $r = 0$ on the magnetic axis.
2. The proper toroidal angle ϕ
3. A poloidal coordinate θ that is periodic, like an angle.

There are many possibilities:



poloidal angle / straight fieldline coordinates / orthogonal coordinates

In straight field line coordinates, the field lines are given by $d\phi/d\theta = q(r)$.

Large aspect ratio expansion

- In a thin torus, or in the plasma center, the inequality $r \ll R$ can be exploited.
- Locally, a thin torus resembles a cylinder
- deviations from circular cross sections are small
- Due to the pressure, these circles are shifted outward slightly.

⇒ Each surface has a Shafranov shift $\Delta(\psi)$

Use the proper (minor) radius r as coordinate.

$$R = R_0 + r \cos \theta + \Delta(r)$$

$$Z = r \sin \theta$$

- Radial force balance:

$$\frac{1}{2r^2} (r^2 \psi'^2)' + R_0^2 p' + FF' = 0.$$

Shafranov shift

Asymmetric part of the force balance ($\sim \cos \theta$):

$$\Delta'' + \left(2\frac{\psi''}{\psi'} + \frac{1}{r}\right)\Delta' - \frac{1}{R_0} + 2\frac{rR_0p'}{\psi'^2} = 0$$

Integrate \longrightarrow outward Shafranov shift given by

$$\Delta'(r) = \frac{r}{R_0} \left(\beta_p(r) + \frac{1}{2}\ell_i(r)\right)$$

in terms of the normalized **internal inductance**

$$\ell_i(r_0) = 2\frac{q^2}{r_0^4} \int_0^{r_0} \frac{r^3}{q^2} dr$$

and the **poloidal beta**,

$$\beta_p(r_0) \equiv \frac{\langle p \rangle_{r < r_0}}{\langle \frac{1}{2} B_p^2 \rangle_{r=r_0}} \approx -2\frac{R_0^2 q^2}{B_0^2 r_0^4} \int_0^{r_0} p' r^2 dr.$$

Vertical field

The poloidal field at the plasma edge depends on θ :

$$B_p = -\frac{\psi'}{R_0} \left[1 + \frac{a}{R_0} \left(\beta_p + \frac{1}{2} \ell_i - 1 \right) \cos \theta \right].$$

⇒ External magnetic measurements can determine $\beta_p + \frac{1}{2} \ell_i$.

Note: Only in a non-circular tokamak, the external measurements can determine β_p and ℓ_i separately.

This $B_p(\theta)$ is determined by plasma equations, but cannot be generated by the plasma current alone.

An external field is required, which, at some distance, is a vertical field

$$B_z = \frac{I_p}{4\pi R} \left(\log(8R/a) + \beta_p + \frac{1}{2} \ell_i - \frac{3}{2} \right).$$