MHD Instabilities in Tokamaks

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Outline:

- Intro: the stability problem
- Ideal MHD
- Linear modes
- Energy principle
- current-driven instabilities
- pressure-driven instabilities

Why expect tokamak instabilities?

Produce fusion energy at lowest cost:

- Maximize density (high ∇n)
- Maximize temperature (high ∇T)
- Maximize plasma current (high magnetic energy)
- confine fusion α-particles ($E_\alpha \gg T_i$)
- Heat the plasma ($T_e \neq T_i$, $T_\perp \neq T_\|$)

$\Longrightarrow$ Instability

- Large / small volume
- large / small amplitude
- fast / slow growth
- δn, δT, $\delta \boldsymbol{B}$, and/or δn_α.

$\Longrightarrow$ limits or reduces ∇n, ∇T, I_{plasma}, and/or $n_\alpha E_\alpha$.

What sort of instabilities?

Instabilities can be caused by

- suprathermal particles (e.g. fusion α's)

$\Longrightarrow$ rapid plasma oscillations, fast particle loss.

(Alfvén-Eigenmodes, fishbones ($q = m/n = 1/1$))

- steep pressure or temperature gradients

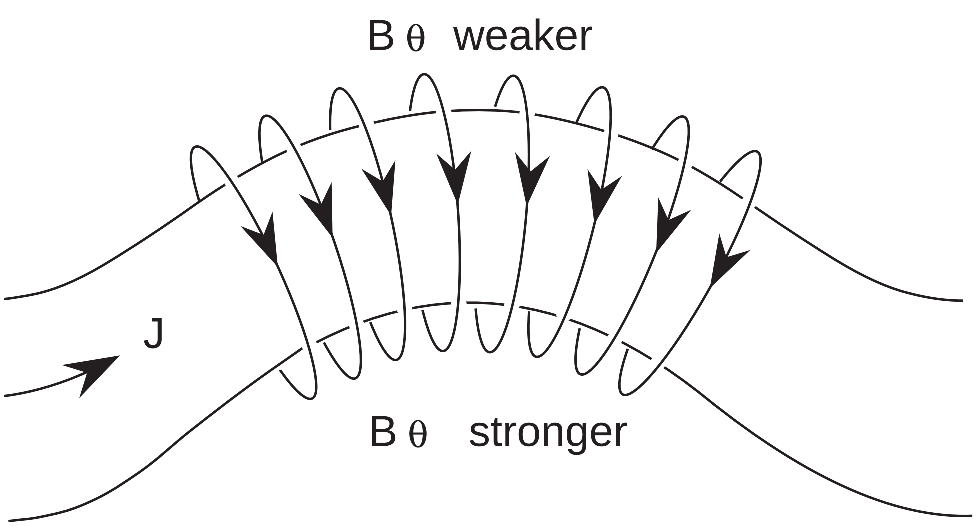
$\Longrightarrow$ small-scale convection, turbulence, heat loss, edge localized modes

(high-n ballooning, ITG (∇T_i), ETG (∇T_e), trapped electron modes)

- high plasma current density

$\Longrightarrow$ large-scale motion, magnetic islands, plasma disruption

(Internal kink ($q = 1$, sawtooth), tearing mode ($q = 3/2,\ 2$), external kink)



The stability problem

Consider a plasma equilibrium (measured or computed)

- Predict if it is stable or unstable.
- Describe the predicted instabilities
  (to compare with instabilities observed in experiments)

Applications

- Design a magnetic confinement experiment
  (magnetic system, control of plasma pressure, current density, rotation)
- Fast feedback to suppress sawteeth, NTMs, disruptions
- Keep bursty instabilities (sawteeth, ELMs) small
- Diagnose the plasma, e.g.: TAE-spectrum $\longrightarrow$ current profile, $q(r)$.
- Understand/predict turbulent fluctuation levels.

Two basic approaches:

1. Nonlinear (numerical) modelling of plasma evolution

$\Longrightarrow$ shows growing instabilities

$\Longrightarrow$ long-term fate: saturation / other instabilities / turbulence

2. Linear perturbations of equilibrium (This lecture)

$\Longrightarrow$ stable perturbations (wave-like, oscillating)

$\Longrightarrow$ unstable perturbations (exponentially growing)

$\Longrightarrow$ Stability boundaries in the equilibrium parameters

$\Longrightarrow$ Eigenfunctions

- Spatial structure
- relative importance δn, δT, $\delta \boldsymbol{B}$

Resistive MHD

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{U})\,,$$

$$\frac{\partial p}{\partial t} = -\boldsymbol{U}\cdot\boldsymbol{\nabla}p - \gamma p\boldsymbol{\nabla}\cdot\boldsymbol{U}\,, \qquad \gamma = 5/3\,,$$

$$\rho\Big(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U}\Big) = \boldsymbol{j}\times\boldsymbol{B} - \nabla p\,,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla}\times\boldsymbol{E}\,,$$

$$\boldsymbol{E} = \eta\boldsymbol{j} - \boldsymbol{U}\times\boldsymbol{B}\,,$$

$$\boldsymbol{j} = \boldsymbol{\nabla}\times\boldsymbol{B}\,,$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0)\,.$$

Ideal MHD

- Hot plasma: negligible resistivity $\eta \sim T^{-3/2} \approx 0$

$$\rho \frac{d\boldsymbol{U}}{dt} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{\nabla}\left(p + \tfrac{1}{2}B^2\right),$$

$$\frac{ds}{dt} = 0\,,$$

$$\frac{d\rho}{dt} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{U}\,,$$

$$\frac{d\boldsymbol{B}}{dt} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{U} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{U}\,.$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\,,$$

$$s \equiv \frac{p}{\rho^\gamma}\,.$$

Ideal MHD (Conservation laws)

$$\rho \frac{d\boldsymbol{U}}{dt} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B} - \boldsymbol{\nabla} \left(s\rho^{\gamma} + \tfrac{1}{2} B^2 \right) ,$$

$$\frac{ds}{dt} = 0 ,$$

$$\frac{d}{dt} \int \rho \, d^3 x = 0 ,$$

$$\frac{d}{dt} \int \boldsymbol{B} \, d^2 \boldsymbol{x} = 0 .$$

$\Longrightarrow$ Magnetic field lines...

- move with the plasma flow
- cannot be created or annihilated
- cannot break up and reconnect
- cannot pass through each other

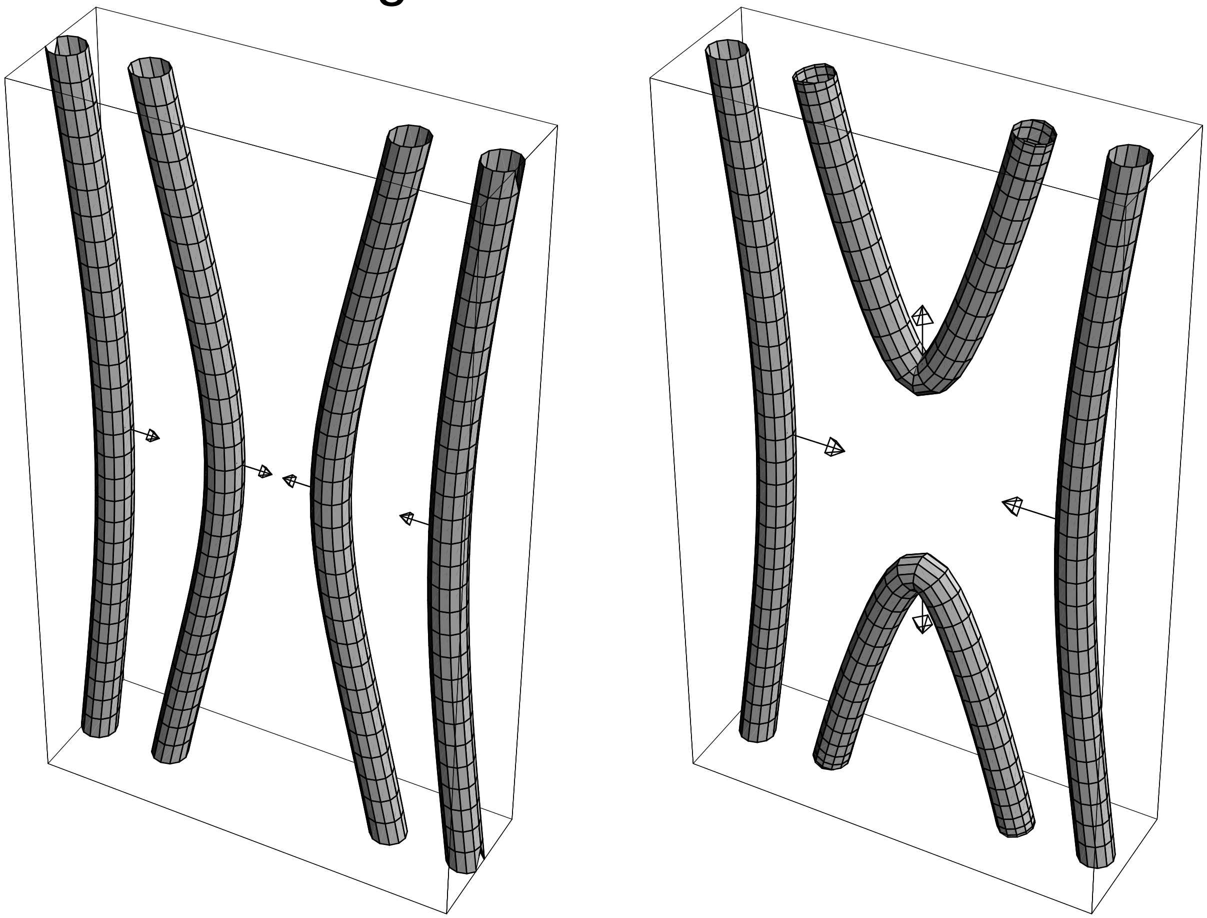
$\Longrightarrow$ Magnetic topology is conserved, flux surfaces remain closed.

What is magnetic reconnection?

- The break-up and merging of magnetic field lines in a plasma.
- Significance:

  Plasma particles move are tied to the field lines

  Reconnection $\Longrightarrow$ loss of magnetic confinement

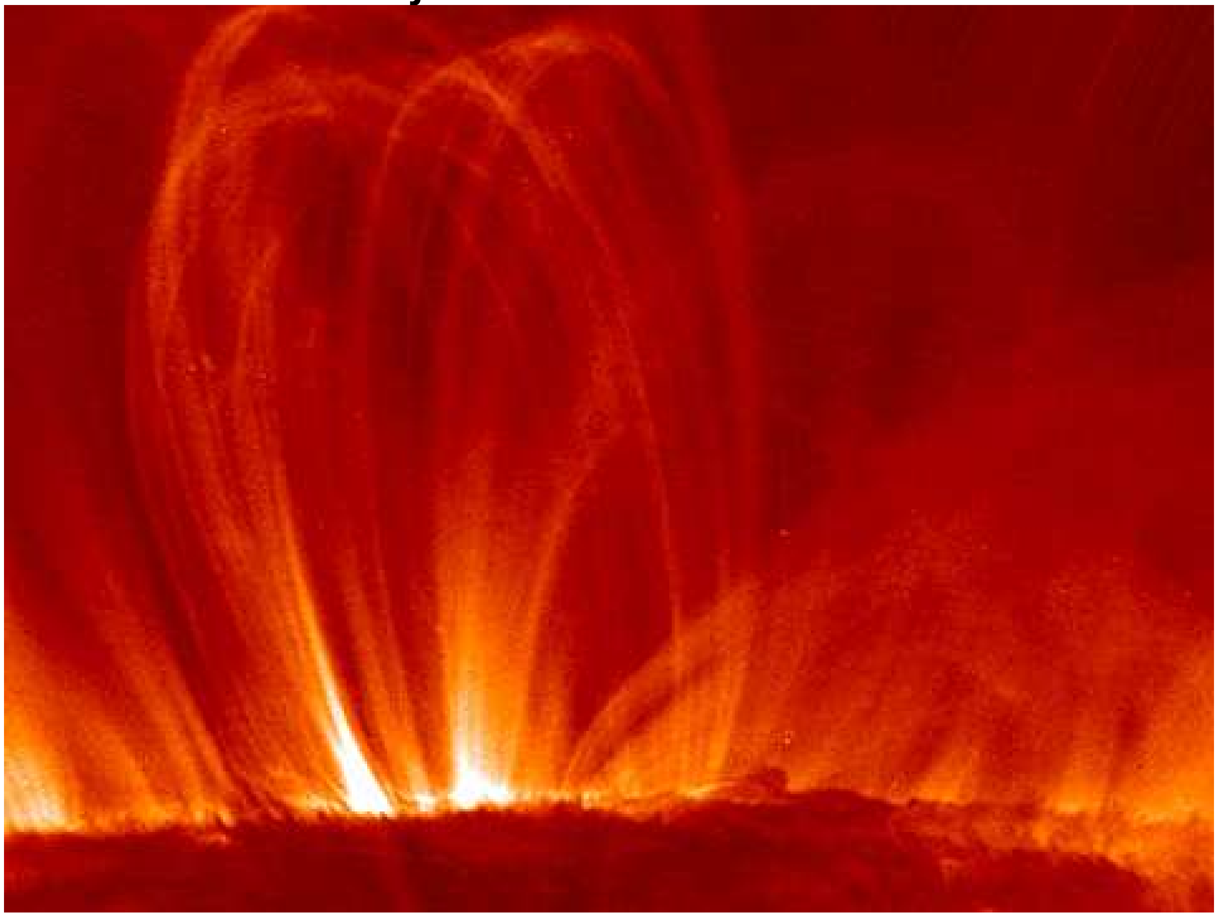


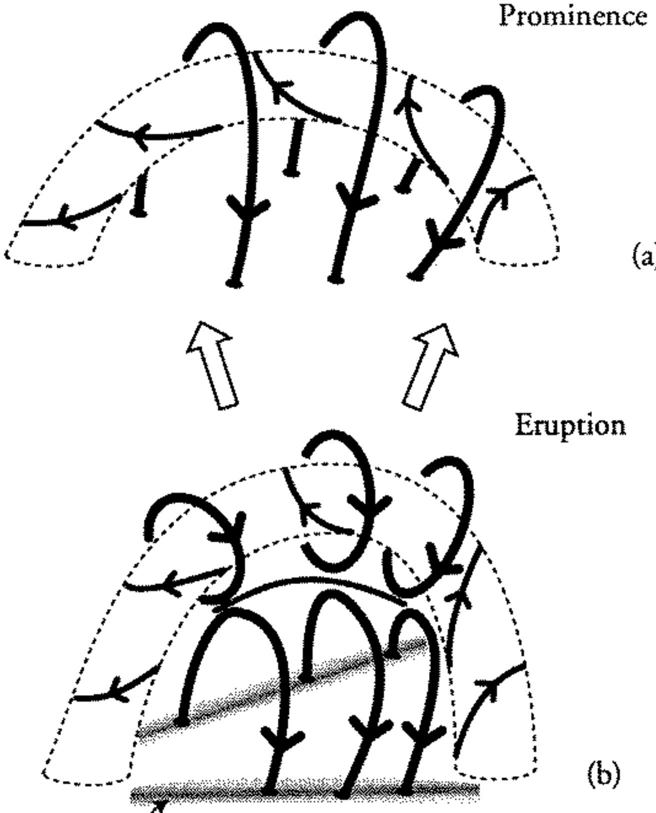
Solar corona:

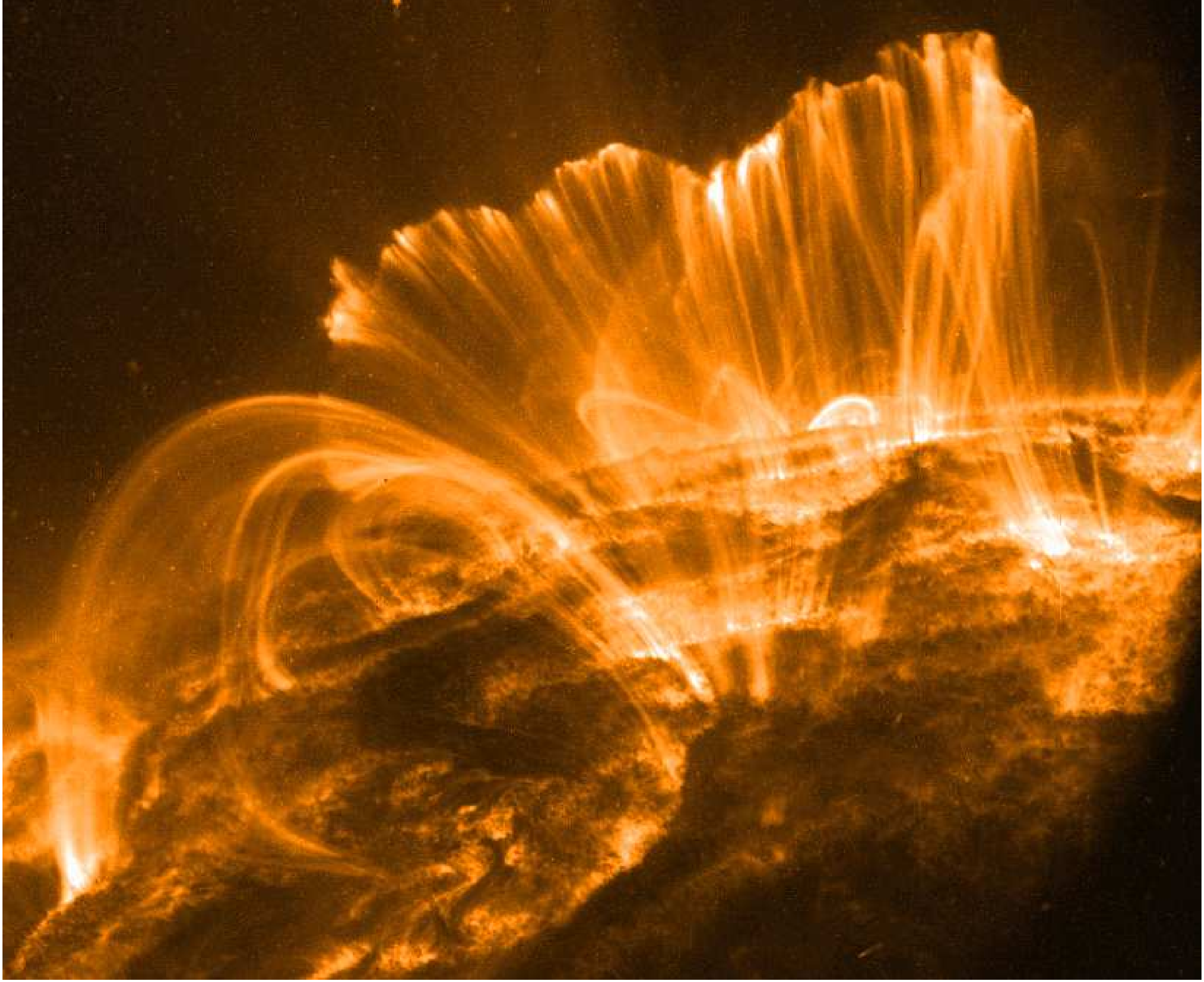
- Magnetic loops with foot points in solar atmosphere
- Reconnection $\Longrightarrow$ open field lines

$\Longrightarrow$ Solar flares

$\Longrightarrow$ Coronal mass ejections



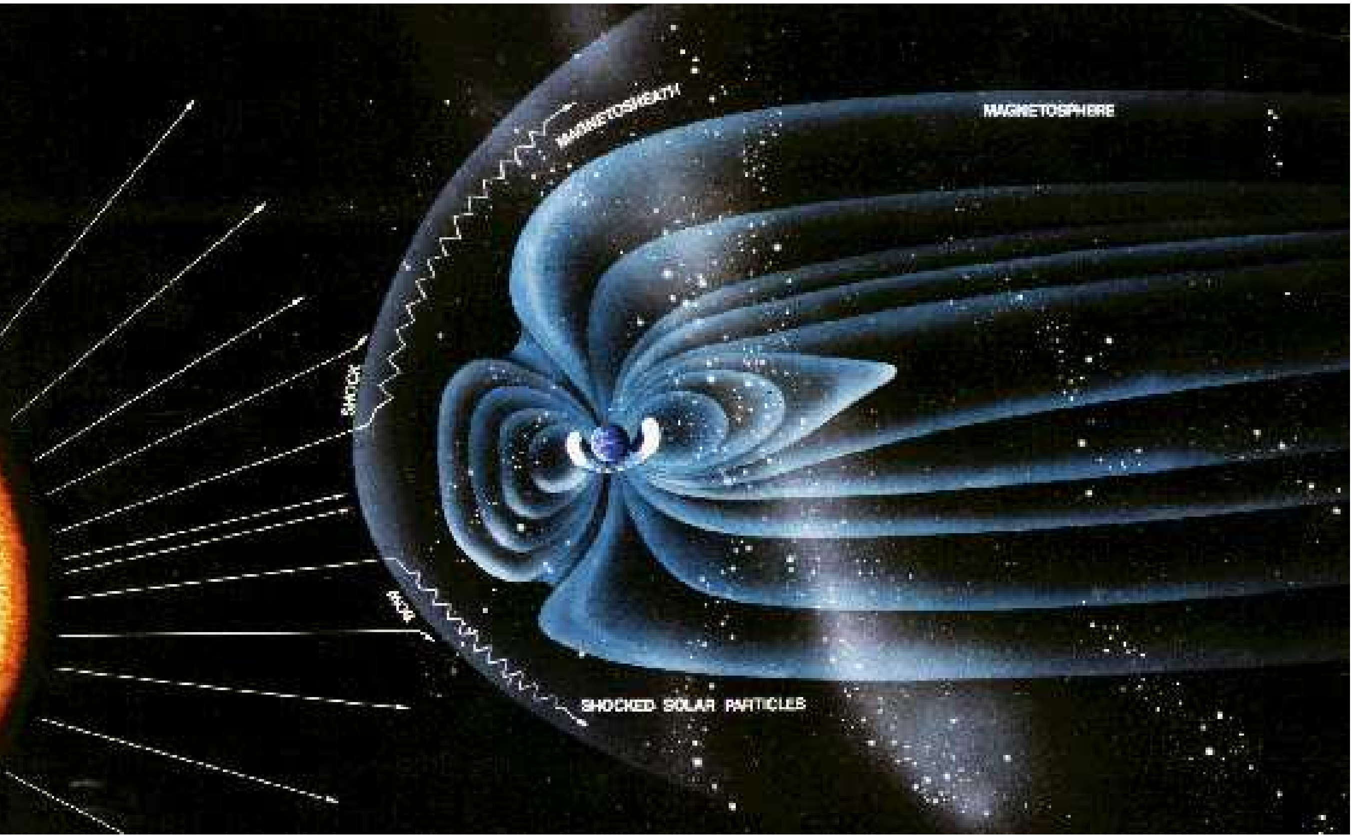


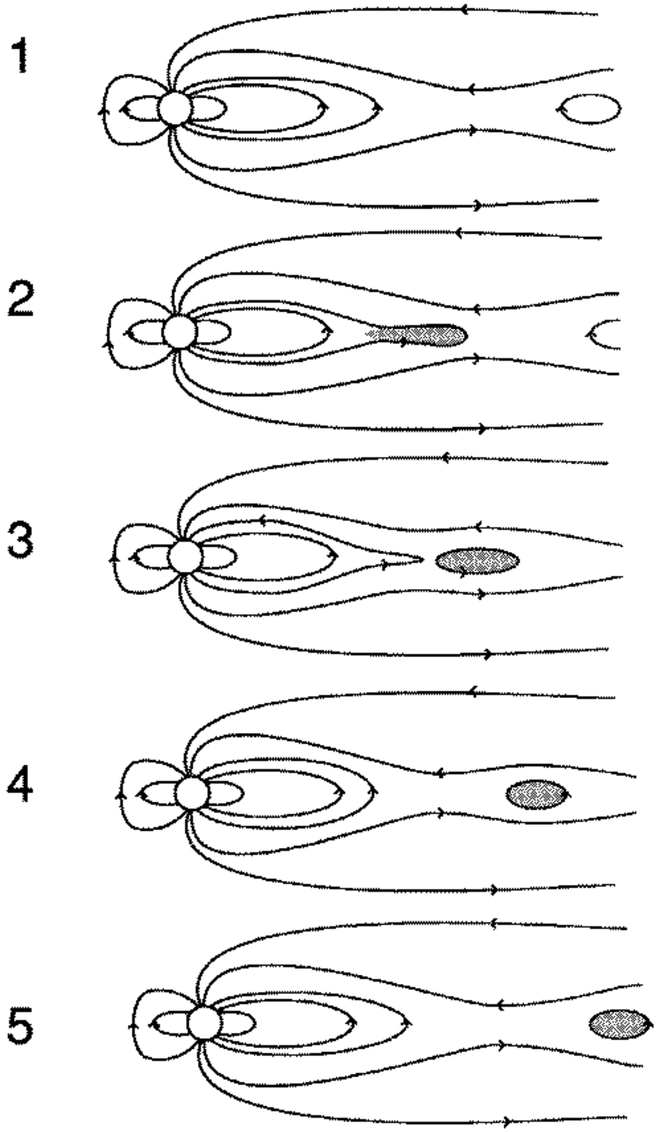


Earth magneto-tail

- Earth magnetic field interacts with solar wind

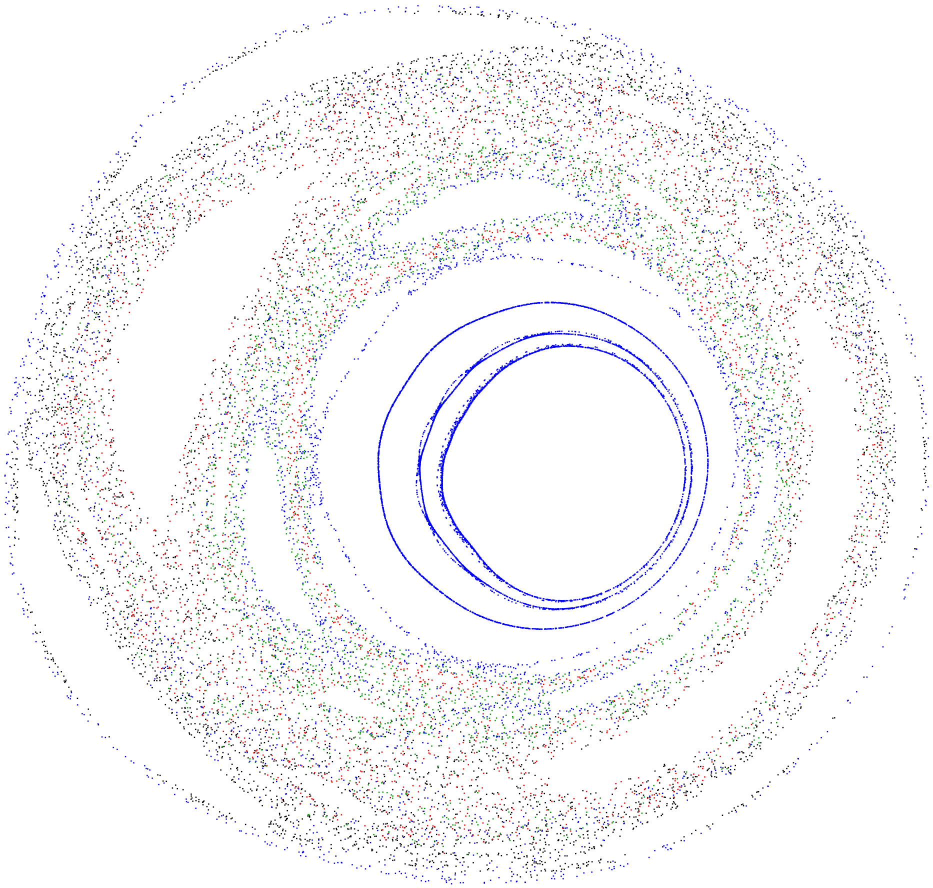
⟹ Magnetic substorms in the magneto-tail





Reconnection can destroy tokamak confinement

Reconnection can turn nested magnetic surfaces into this:



⟹ Release of heat and magnetic energy

What is special about reconnection?

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  - Regions of vanishing field
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- This current sheet screens the plasma from the magnetic perturbation

$\Longrightarrow$ No reconnection

When is reconnection interesting?

- When magnetic topology constrains the plasma dynamics...
  It can confine magnetic energy, thermal energy, particles.
  (In a magnetized plasma $\chi_{\|} \gg \chi_{\perp}$).

- Examples:
  - Solar interior (solar cycle)
  - Solar corona (flares)
  - Earth magnetosphere, magneto-tail (magn. substorms)
  - magnetically confined fusion plasmas (disruptions)

- Where?
  - Magnetic nulls
  - Magnetic resonances (with a strong magnetic guide field)

Local reconnection process can have global transport effects

Mechanisms for magnetic reconnection

- Reconnection can take place when:
  - Formation of a current sheet is delayed / slowed down
  - Formation of a current sheet is not complete
  - The current sheet forms but subsequently decays

- In detail:
  - slow down or limit the acceleration of electrons in the current sheet.

- Challenge:

  Often reconnection is observed to proceed faster than can be explained with straightforward mechansims.

Specific mechanisms:

- collisional friction on electrons
  (resistive decay of current sheet if plasma is collisional)

- electron inertia
  (delays the current sheet formation)

- Landau damping
  (wave-particle resonances remove the electron energy)

Example:

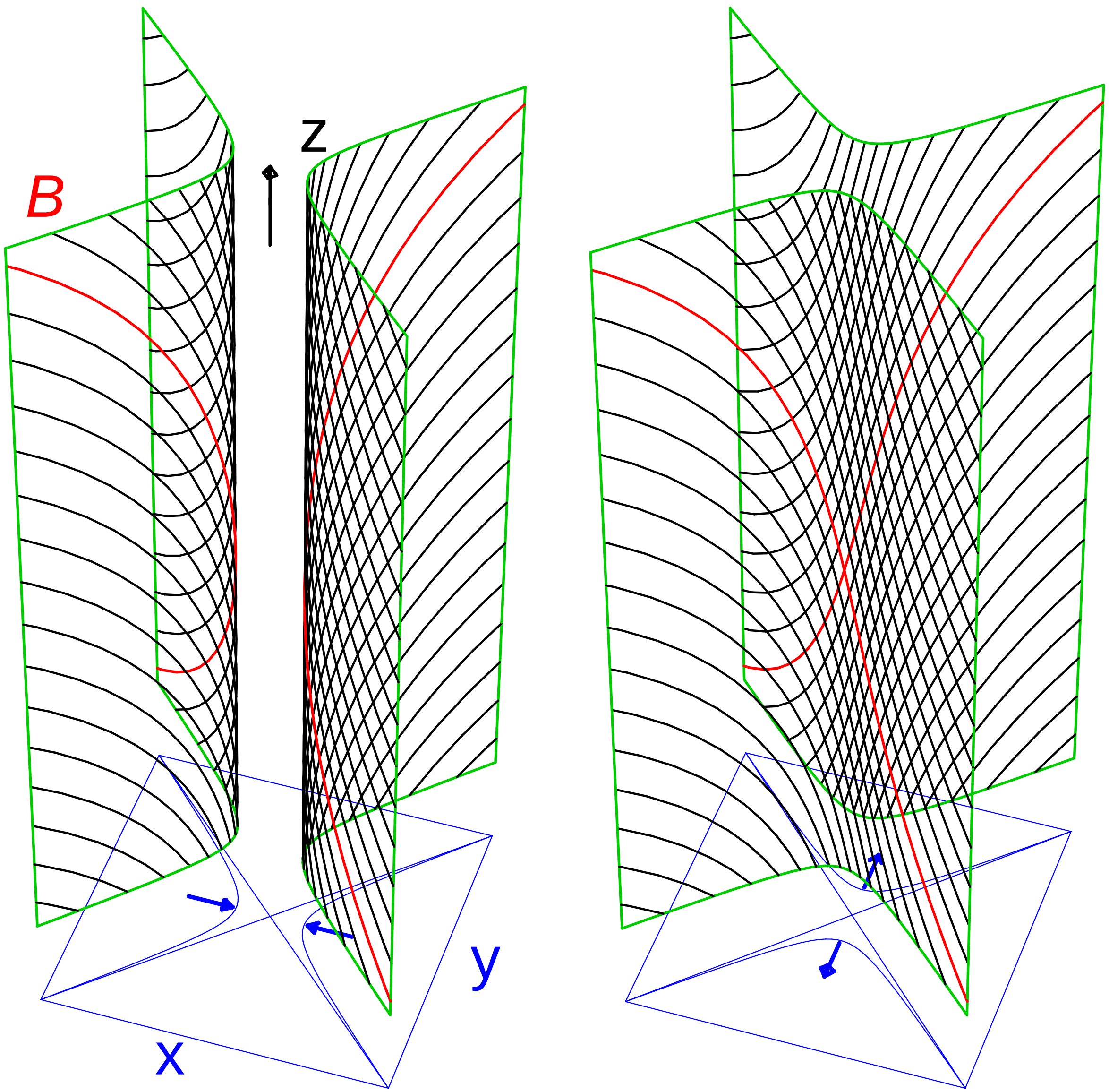
Fast ($\sim 30\,\mu$s) sawtooth crash of $T_e(r=0)$ (internal disruptions) in JET
$\Longrightarrow$ Negligible resistivity $\Longrightarrow$ 'Collisionless' reconnection

- Electron inertia provides fast mechanism

(1) Numerical demonstration: isothermal 2-fluid model
(2) Collisionless electrons $+\nabla T_e \Longrightarrow$ kinetic model necessary.

Strong magnetic guide field

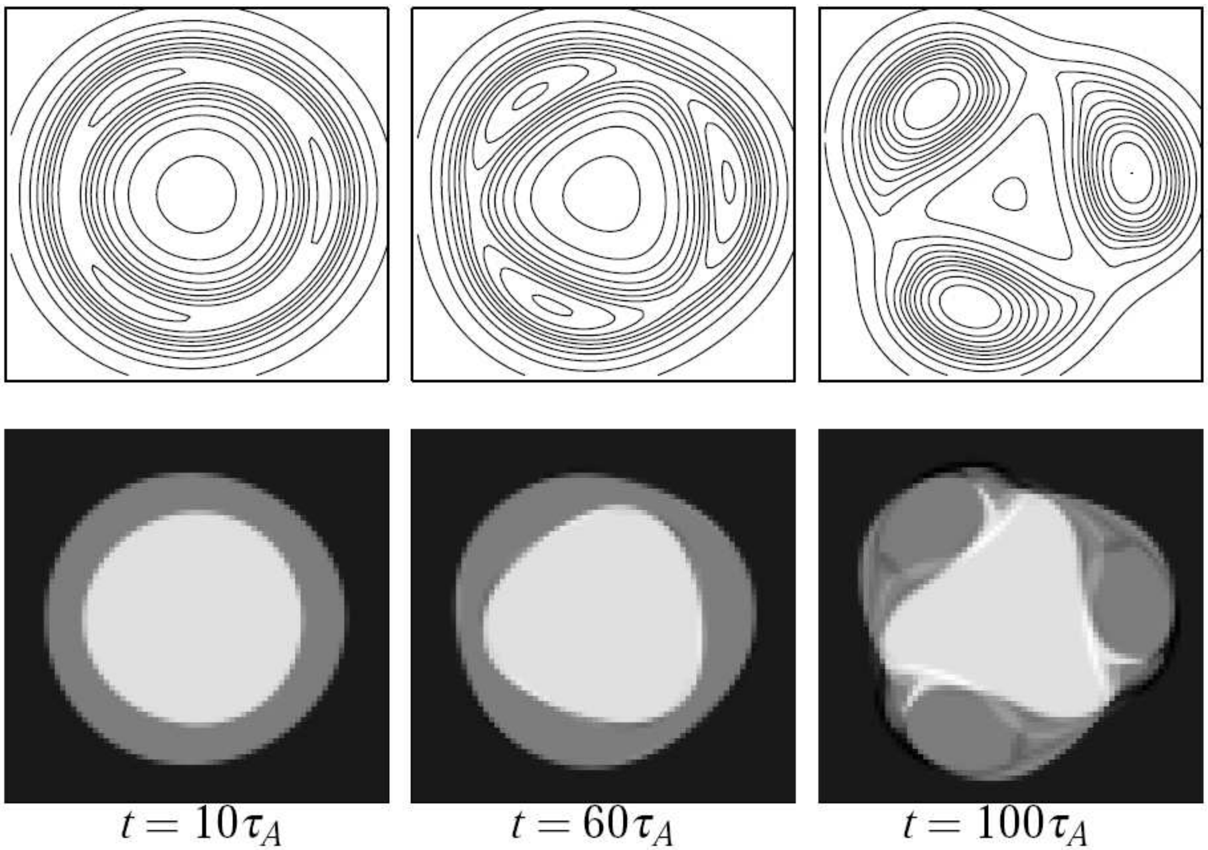
- reconnection at:
  - magnetic resonance
  - straight resonant zone



Tearing mode instability

- The instability causes reconnection at $x = 0$.
- A chain of magnetic islands is formed.
- Such an instability is called tearing mode.

Example: $m = 3$ mode in case of low magnetic shear (small dq/dr)
(top: ψ, bottom T_e)



Hamilton's principle

The plasma motion $\boldsymbol{U}(\boldsymbol{x}, t)$ makes the action

$$S = \int_{t_0}^{t_1} L \, dt$$

stationary, where the Lagrangian is

$$L = \int d^3x \left(\tfrac{1}{2}\rho U^2 - \frac{p}{\gamma - 1} - \tfrac{1}{2} B^2 \right) .$$

Infinitesimal perturbations

Describe linear modes with an infinitesimal perturbation $\delta_\xi \boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{x}, t)$.

$$\delta_\xi \boldsymbol{U} = \frac{d\boldsymbol{\xi}}{dt},$$

$$\delta_\xi s = 0,$$

$$\delta_\xi \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{\xi},$$

$$\delta_\xi \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{\xi} - \boldsymbol{B} \boldsymbol{\nabla} \cdot \boldsymbol{\xi}.$$

$\Longrightarrow$

$$\delta_\xi L = -\int d^3x \, \boldsymbol{\xi} \cdot \left(\rho \frac{d\boldsymbol{U}}{dt} - \boldsymbol{j} \times \boldsymbol{B} + \nabla p\right) \quad \text{+ boundary terms}$$

Linearized MHD equation

Perturbing the momentum equation

$\Longrightarrow$ equation of motion for $\boldsymbol{\xi}$:

$$0 = \delta_\xi \big(\rho \frac{d\boldsymbol{U}}{dt} - \boldsymbol{j} \times \boldsymbol{B} + \nabla p\big)$$

$$= \rho \frac{d^2 \boldsymbol{\xi}}{dt^2} - \boldsymbol{F}(\boldsymbol{\xi}).$$

The linear force operator is

$$\boldsymbol{F}(\boldsymbol{\xi}) = (\boldsymbol{\nabla} \times \boldsymbol{Q}) \times \boldsymbol{B} + (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{Q}$$

$$+ \boldsymbol{\nabla}(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p + \gamma p \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + \boldsymbol{\nabla} \cdot \big(\rho \boldsymbol{\xi} \frac{d\boldsymbol{U}}{dt}\big)$$

and $\boldsymbol{Q} \equiv \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})$.

The force operator F is self-adjoint

$$\int \boldsymbol{\zeta} \cdot \boldsymbol{F}(\boldsymbol{\xi}) d^3x = \int \boldsymbol{\xi} \cdot \boldsymbol{F}(\boldsymbol{\zeta}) d^3x$$

Outline of the proof:

$$\delta_\zeta \delta_x i L = \int \rho \frac{d\boldsymbol{\zeta}}{dt} \cdot \frac{d\boldsymbol{\xi}}{dt}\, d^3x + \int \boldsymbol{\zeta} \cdot \boldsymbol{F}(\boldsymbol{\xi})\, d^3x$$

$$\equiv 2K(\boldsymbol{\zeta}, \boldsymbol{\xi}) - 2\delta W(\boldsymbol{\zeta}, \boldsymbol{\xi})$$

Infinitesimal displacements are a Lie algebra $(\delta_\zeta \delta_\xi - \delta_\xi \delta_\zeta = \delta_\eta)$

$\Longrightarrow \delta_\zeta \delta_\xi L = \delta_\xi \delta_\zeta L$

$\Longrightarrow \delta W(\boldsymbol{\zeta}, \boldsymbol{\xi}) = \delta W(\boldsymbol{\xi}, \boldsymbol{\zeta})$

Consequences of self-adjointness

There are normal mode solutions $\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{\xi}(\boldsymbol{x})e^{-i\omega t}$ with

$$-\rho\omega^2\boldsymbol{\xi} = \boldsymbol{F}(\boldsymbol{\xi})$$

and $\omega^2 =$ real.

$\Longrightarrow$ ω is real (wave) or imaginary (instability)

$\Longrightarrow$ A marginally stable ideal MHD mode has $\omega = 0$.

$\Longrightarrow$ Only non-MHD effects can turn a wave with $\omega^2 \gg 0$ into an instability.
Example: High-energy ions (fusion α-particles) can make a high-frequency Alfvén eigenmode (‘TAE-mode’) unstable.

Without such non-ideal effects, ‘High frequency’ means ‘very stable’.

The Energy Principle

An equilibrium is (ideal MHD) stable if and only if

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) \geq 0$$

for all possible displacements $\boldsymbol{\xi}$, which satisfy appropriate boundary conditions and are bounded in energy.

- Application:
  Finding a function $\boldsymbol{\xi}$ with $\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) < 0$ is enough to prove instability even if $\boldsymbol{\xi}$ is not a solution.
  i.e., one does not need to know the precise instability.

Marginal stability

Stability limits can be found by minimizing $\delta W(\boldsymbol{\xi})$.

$\Longrightarrow$ $\boldsymbol{F}(\boldsymbol{\xi}) = 0$. Advantages:

- Systematically close in on the most unstable modes
- Obtain accurate stability boundaries even with less accurate mode solutions (inaccurate eigenfunctions)

Intuitive form of the energy functional

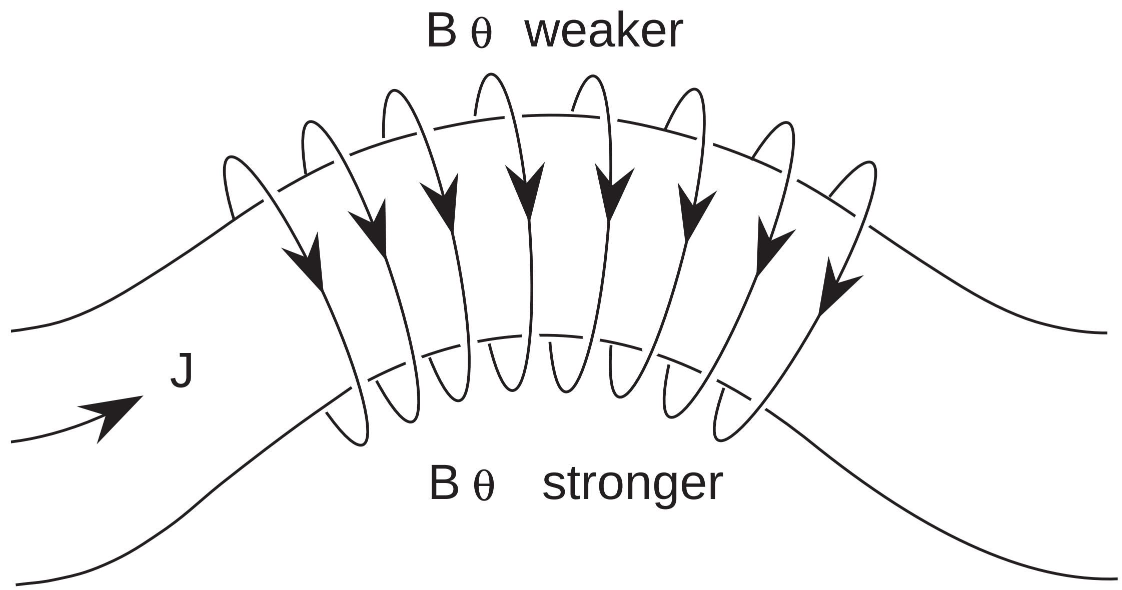
$$\delta W = \frac{1}{2}\int_P d^3x \Big[\gamma p|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2 + |\boldsymbol{Q}_\perp|^2 + B^2|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2$$

$$- j_\parallel(\boldsymbol{\xi}_\perp^* \times \boldsymbol{b}\cdot\boldsymbol{Q}_\perp) - 2(\boldsymbol{\xi}_\perp\cdot\nabla p)(\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa})\Big].$$

- stabilizing terms
  1. avoid plasma compression, $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$
  2. avoid fieldline bending, $\boldsymbol{B}\cdot\boldsymbol{\nabla} = 0$
  3. avoid field compression, $\delta B = 0$
- current driven instabilities
- pressure driven instabilities

Current-driven instabilities

A current-carrying wire can kink due to self-forces.



- The most unstable modes have long wavelengths
- mode $\sim \exp i(m\theta - n\phi)$
- Small $\boldsymbol{B}\cdot\boldsymbol{\nabla} \Longrightarrow m/n \approx q$ (magnetic resonance)

Ideal MHD stability analysis:

$\Longrightarrow$ Internal kink modes stable except $m = n = 1$ for $\beta_p > \beta_{\rm crit}$ (if the $q = 1$ surface exists)

$\Longrightarrow$ External kink modes unstable if:

- m is small, and the $q = m/n$-surface is near the edge
- There is no stabilizing conducting wall near the plasma.

Pressure-driven instabilities

The term $-2(\boldsymbol{\xi}_\perp \cdot \nabla p)(\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa})$ is destabilizing where ∇p and $\boldsymbol{\kappa}$ are in the same direction.

$\Longrightarrow$ Instability balloons at low-field-side of tokamak

- $\boldsymbol{B} \cdot \boldsymbol{\nabla}$ is not as small as in kink modes
- Most unstable for high mode number n

$\Longrightarrow$ Stability analysis can be done for $n \rightarrow \infty$

- Radially localized modes

$\Longrightarrow$ Stability criterium $|\nabla p| < C_{\text{crit}}(r)$

- The pressure limit depends on the magnetic shear $\frac{r}{q}\frac{dq}{dr}$.

Second stability

Ballooning stability actually depends on the local shear $|d\boldsymbol{B}/dr|$

The local shear varies significantly over a flux surface. This variation is caused by:

- The Shafranov shift
- Elongated plasma cross-section
- Triangular or D-shape

At very high pressures (above the stability boundary), the Shafranov shift can create a second region of stability.

At high triangularity, this “second stability” is possible at lower pressure gradients, provided the shear is negative.

Troyon Limit

A classic example of how to optimize MHD equilibrium to avoid several instabilities was given by F. Troyon et al., in 1984. Optimize the total plasma β by varying

- pressure profile $p(\psi)$
- current profile, $q(\psi)$
- plasma shape (ellipticity, triangularity)

Subject to the constraints

- No sawteeth and interchange modes: $q > 1$.
- No external knik modes: $q(a) > 2$.
- Everywhere stable to local ballooning modes

$$\Longrightarrow \quad \beta < 0.028 \frac{I_{\rm plasma}}{aB_t}.$$

This leads to the definition of $\beta_N \equiv \beta\, aB_t / I_{\rm plasma}$.

Negative shear and plasma rotation give improvements over Troyon's limit.