Exercises 5,6 Plasma Physics, Hand in by 29 October 2009

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a. Estimate the magnetic Reynolds number for two typical plasmas listed in the table on page 11: the Solar corona with a typical magnetic field strength of 10^{-8} T and an experimental fusion plasma with a magnetic field strength of 5 T. Assume that the relevant length scales are 10^8 m and 1 m respectively. ssume that in both cases, the electric resistivity is comparable with that of copper: $2 \times 10^{-8}\Omega$m.
In both cases, we need an estimate of a typical velocity of the magneto-hydrodynamic motion. You can make a rough estimate by comparing the inertial term $\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in the equation of motion with either the pressure gradient term ∇p or the magnetic force $\boldsymbol{B} \times \boldsymbol{J}$.
Show that if the comparison with ∇p is made, your velocity estimate is the sound speed or the thermal velocity of the ions.
Now focus on the second comparison, between the inertial term and the magnetic force. Rewrite the magnetic force so that the gradient of the magnetic pressure appears: $\nabla(B^2/2\mu_0)$. Based on this comparison, show that typical magneto-hydrodynamic velocities are comparable to the so-called Alfvén speed $B/(\mu_0 \rho)^{1/2}$. Compute the Alfvén speed for the two example plasmas at hand and use these numbers to estimate the magnetic Reynolds number.

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A mathematical theorem states that any divergence-free vector field $\boldsymbol{B}(\boldsymbol{x}, t)$ can be expressed in terms of two scalar fields $\alpha(\boldsymbol{x}, t)$ en $\beta(\boldsymbol{x}, t)$ as

$$\boldsymbol{B} = \nabla \alpha \times \nabla \beta.$$

a. Show that $\boldsymbol{B}$, written this way, is indeed divergence-free.
b. Give a geometric interpretation of the field lines of $\boldsymbol{B}$ in terms of the surfaces given by α=constant and β=constant.
c. Assume that the time evolution of α and β are determined by the flow field $\boldsymbol{U}(\boldsymbol{x}, t)$ according to the equations

$$\frac{\partial \alpha}{\partial t} + \boldsymbol{U} \cdot \nabla \alpha = 0, \qquad \frac{\partial \beta}{\partial t} + \boldsymbol{U} \cdot \nabla \beta = 0.$$

Explain what these equations mean, geometrically.
d. From all the above expressions, derive an equation that gives $\partial \boldsymbol{B}/\partial t$ in terms of $\boldsymbol{B}$ and $\boldsymbol{U}$ instead of α and β.
e. Using this analysis, show that the topology of $\boldsymbol{B}$-fieldlines cannot change.