exercise: Magnetic islands — hand in by 14 January 2010

An axisymmetric magnetic field can always be written as

$$\boldsymbol{B} = \nabla\phi \times \nabla\psi + F(\psi)\nabla\phi,$$

where ϕ is the toroidal angle and $\nabla\phi \cdot \nabla\psi = 0$.

a. Show that $\nabla \cdot \boldsymbol{B} = 0$ and the field lines lie in the surfaces given by $\psi =$ constant. Show that on any given magnetic surface the toroidal field component is inversely proportional to the distance to the symmetry axis.

b. We now build a simplified, “straight” model of the magnetic surfaces that imitate a torus with minor radius r and major radius R. To this end we introduce cartesian coordinates (x, y, z): x is de radial coordinate, y is the poloidal coordinate (periodic for $y \to y + 2\pi r$), and z is the toroidal coordinate (periodic for $z \to z + 2\pi R$). The axisymmetric field becomes

$$\boldsymbol{B} = (\hat{\boldsymbol{z}} \times \nabla\psi + F(x)\hat{\boldsymbol{z}})/R$$

This simplifies to $\boldsymbol{B} = (F(x)\hat{\boldsymbol{z}} - (d\psi/dx)\hat{\boldsymbol{y}})/R$ because $\psi = \psi(x)$. Show that the safety factor is $q(x) = rF/(R\,d\psi/dx)$. Compute the current density $\boldsymbol{J}$.

c. Consider the special case $F(x) = B_0R$ and $\psi(x) = \frac{1}{2}J_0\mu_0Rx^2$. Show that in this case the current density is constant. Compute the safety factor profile $q(x)$ and the magnetic shear $s \equiv (dq/dx)x/q$.

d. Take once more $F(x) = B_0R$ so that the magnetic field has the form

$$\boldsymbol{B} = \hat{\boldsymbol{z}} \times \nabla\psi/R + B_0\hat{\boldsymbol{z}}.$$

However, now introduce a helical symmetric perturbation with poloidal wave number m and toroidal wave number n by introducing y, z dependencies in ψ:

$$\psi = \frac{1}{2}J_0\mu_0Rx^2 + \psi_1\cos\Big(\frac{m}{r}y - \frac{n}{R}z\Big).$$

Show that still $\nabla \cdot \boldsymbol{B} = 0$, but that ψ=constant no longer determines a magnetic surface.

e. It turns out, that magnetic surfaces are given by ψ^*=constant with

$$\psi^* = \frac{1}{2}J_0\mu_0R(x - x_r)^2 + \psi_1\cos\Big(\frac{m}{r}y - \frac{n}{R}z\Big).$$

for a suitable value of x_r. Determine this so-called resonance radius x_r and show that ψ^*=constant indeed gives magnetic surfaces. Determine $q(x_r)$.

f. Sketch lines of constant ψ^* in the (x, y)-plane in the neighbourhood of $x = x_r$. Assume for convenience that J_0 and ψ_1 are both positive. What is special about the line(s) $\psi^* = \psi_1$? What is special about the point(s) $\psi^* = -\psi_1$?

g. What is the maximal distance in the x-direction between the $\psi^* = \psi_1$ lines?

h. Give the current density in the z-direction including perturbation. For which value of z is the current density maximal? For which value of z is it minimal?