Exercise: Dust-Acoustic-Wave

During the lecture you learned about the coupling between the discharge and the dust particles via the charging process (= loss of plasma) and quasi-neutrality. In this exercise we will investigate the propagation of a wave through a cloud of charged dust grains. We will look at the propagation of electrostatic waves, in an un-magnetized plasma. The wave propagates so slow that the ions and electrons are able to provide quasi-neutrality during the fluctuations of the dust density. The charge on a dust grain does not change, only the dust density. First we will set up the equations we need.

a) Take equations 109 and 110 from the basics lecture notes and write down the conservation equations for the dust density (no source) and velocity, neglecting the pressure contribution and leaving out the magnetic field.

The equations are:

$$\frac{\partial n_d}{\partial t} + \vec{\nabla} \cdot n_d \vec{v_d} = 0$$

$$m_d n_d \Big(\frac{\partial \vec{v_d}}{\partial t} + \vec{v_d} \cdot \vec{\nabla} \vec{v_d} \Big) = q_d n_d \vec{E}$$

b) The dusty plasma is assumed to be quasi-neutral, with an ion density n_+, an electron density n_e. (The dust has a density n_d and a charge $q_d = eZ_d$, with $Z_d < 0$, usually) Write down the equation for charge neutrality and the Poisson equation for the electric potential in the unperturbed plasma. What is the (trivial) solution if the potential is zero at infinity?

The potential obeys the Poisson equation:

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} \big(n_+ - n_e + n_d Z_d \big) = 0; \lim_{\vec{r} \to \infty} \phi = 0$$

with the trivial solution that ϕ=0 everywhere. This is the potential of the unperturbed system.

c) Now we write the dust density as a uniform density n_{d0} plus a perturbation, n_{d1}, and the same for the dust velocity and the potential: $\vec{v_d} = \vec{v_{d0}} + \vec{v_{d1}}$, $\phi = \phi_0 + \phi_1$. ($\vec{v_{d0}} = 0$ because we consider a static system) Use this to linearize the equations for the dust density and velocity. ($\vec{E} = -\vec{\nabla}\phi$, of course).

Inserting the perturbed quantities and using v_{d0}=0, ϕ_0=0, gives:

$$\frac{\partial (n_{d0} + n_{d1})}{\partial t} + \vec{\nabla} \cdot (n_{d0} + n_{d1}) \vec{v_{d1}} = 0$$

$$m_d (n_{d0} + n_{d1}) \Big(\frac{\partial \vec{v_{d1}}}{\partial t} + \vec{v_{d1}} \cdot \vec{\nabla} \vec{v_{d1}} \Big) = -q_d (n_{d0} + n_{d1}) \vec{\nabla} \phi_1$$

Skipping quadratic contributions and derivatives of the constant time-independent background:

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\vec{\nabla}\cdot\vec{v_{d1}} = 0$$
$$\frac{\partial \vec{v_{d1}}}{\partial t} = -\frac{q_d}{m_d}\vec{\nabla}\phi_1$$

d) For the Poisson equation we assume that both the ion density and the electron density depend on the potential according to the Boltzmann law, with temperatures T_+ and T_e. Write down the Poisson equation for the perturbed plasma and linearize, using first order Taylor expansions when necessary.

The Poisson equation reads:

$$\nabla^2(\phi_0 + \phi_1) = -\frac{e}{\epsilon_0}(n_+ - n_e + Z_d(n_{d0} + n_{d1}))$$

With ϕ_0=0 and Boltzmann and quasi-neutrality for the unperturbed system we have:

$$\begin{aligned}
\nabla^2\phi_1 &= -\frac{e}{\epsilon_0}(n_{+0}\exp(-\frac{e\phi_1}{kT_e}) - n_{e0}\exp(\frac{e\phi_1}{kT_e}) + Z_d(n_{d0} + n_{d1})) \\
&= -\frac{e}{\epsilon_0}(n_{+0}(1 - \frac{e\phi_1}{kT_+}) - n_{e0}(1 + \frac{e\phi_1}{kT_e} + Z_d(n_{d0} + n_{d1})) \\
&= -\frac{e}{\epsilon_0}(-n_{+0}\frac{e\phi_1}{kT_+} - n_{e0}\frac{e\phi_1}{kT_e} + Z_d n_{d1})
\end{aligned}$$

e) Now we keep quasi-neutrality throughout the perturbation, so the right-hand side of the Poisson equation is zero. Use this to derive an expression for the perturbed potential.

Putting the right-hand-side of the perturbed Poisson equation equal to zero gives

$$-n_{+0}\frac{e\phi_1}{kT_+} - n_{e0}\frac{e\phi_1}{kT_e} + Z_d n_{d1} = 0$$
$$(\frac{en_{+0}}{kT_+} + \frac{en_{e0}}{kT_e})\phi_1 = Z_d n_{d1}$$
$$\phi_1 = Z_d n_{d1}/(\frac{en_{+0}}{kT_+} + \frac{en_{e0}}{kT_e}) = Z_d n_{d1} kT_+ kT_e/(en_{+0}kT_e + en_{e0}kT_+)$$

Or, writing the temperature in electronvolts: $kT(K) = eT(eV)$:

$$\phi_1 = Z_d n_{d1} T_+ T_e/(n_{+0}T_e + n_{e0}T_+)$$

Note: we are looking at a wave that is so slow that the ion and electron density fully react to changes in the dust density, maintaining quasi-neutrality (to first order!)

f) Combine the results you have to obtain the dispersion relation (ω^2/k^2) for the wave.

As usual, we assume a periodic perturbation, so $\partial/\partial t$=-iω, $\vec{\nabla}$=i$\vec{k}$.
The equations we have are:

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\vec{\nabla}\cdot\vec{v_{d1}} = 0$$

$$\frac{\partial \vec{v_{d1}}}{\partial t} = -\frac{eZ_d}{m_d}\vec{\nabla}\phi_1$$

$$\phi_1 = Z_d n_{d1} T_+ T_e/(n_{+0}T_e + n_{e0}T_+)$$

Taking the wave vector in the direction of the perturbed velocity, we have:

$$-i\omega n_{d1} + n_{d0} ikv_{d1} = 0$$

$$-i\omega v_{d1} = -ik\frac{eZ_d}{m_d}\phi_1 = -ik\frac{eZ_d^2 n_{d1}T_+T_e}{m_d(n_{+0}T_e + n_{e0}T_+)}$$

Eliminating either n_{d1} or v_{d1} gives the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{eZ_d^2 n_{d0}T_+T_e}{m_d(n_{+0}T_e + n_{e0}T_+)}$$